Strain-stiffening of athermal floppy networks

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acknowledgments

Matthieu Wyart

EPFL

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Eran Bouchbinder

מכון ויצמן למדע
WEIZMANN INSTITUTE OF SCIENCE

strain stiffening

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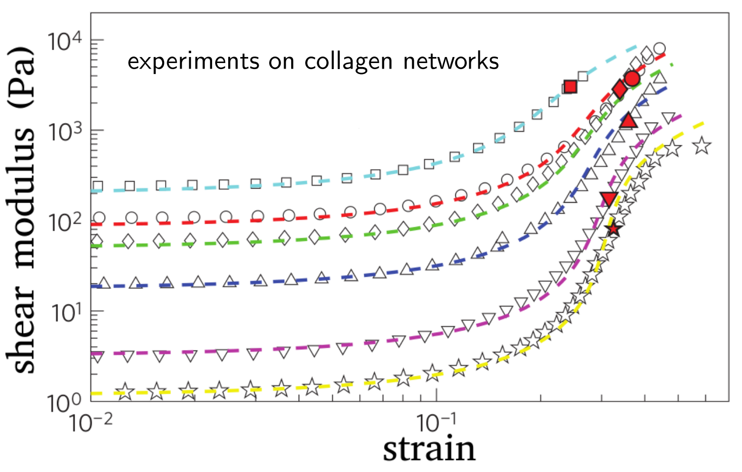

Sharma et al., Nature Physics **12**, 584 (2016)

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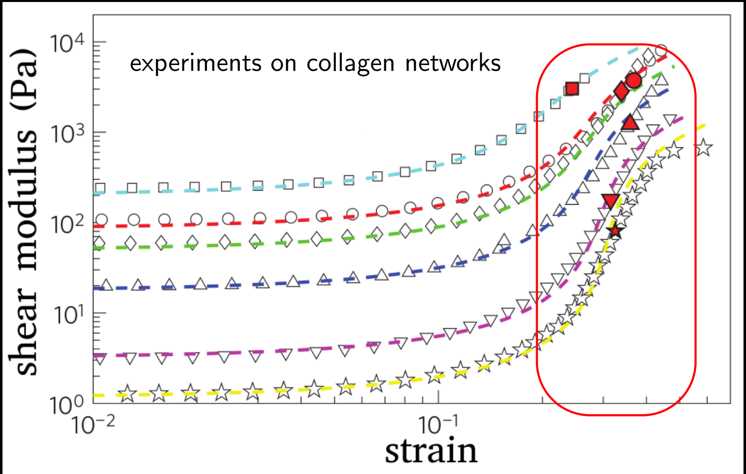


today:

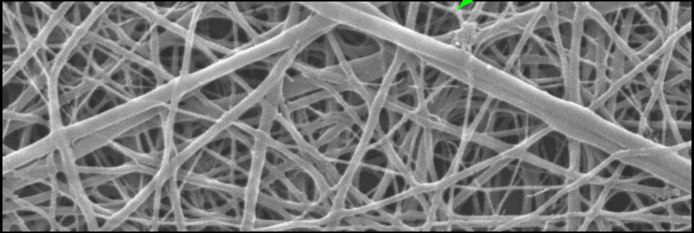
why?
how?

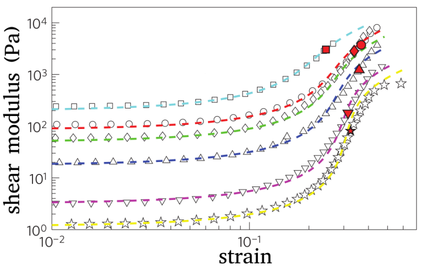
Sharma et al., Nature Physics **12**, 584 (2016)

strain stiffening is the deformation-induced increase in a material's elastic modulus

fibers that are **easy to bend** but **hard to stretch**

collagen networks





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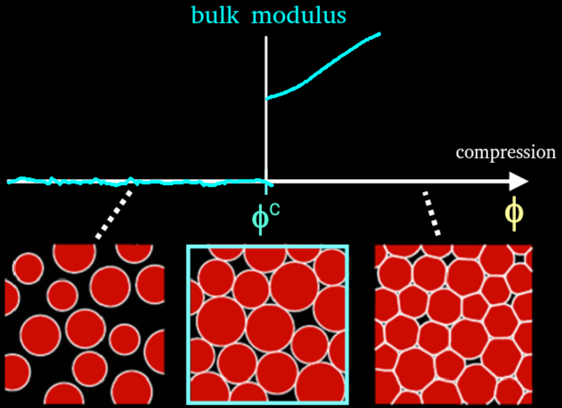

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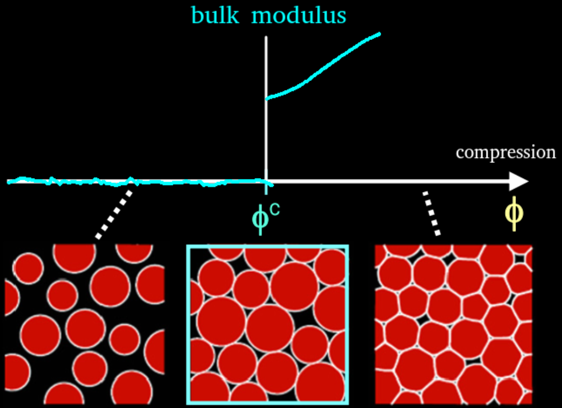
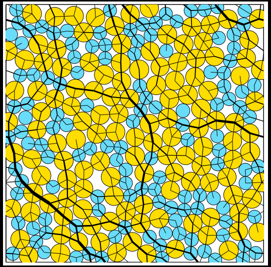
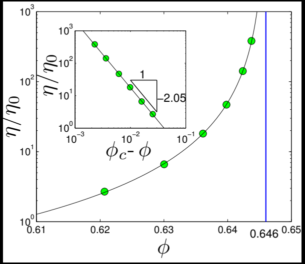


non-Brownian suspension viscosity





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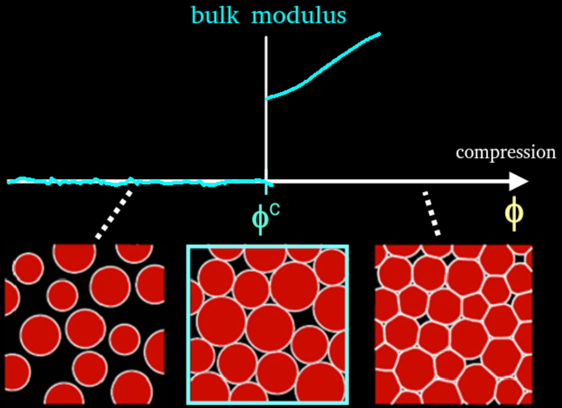

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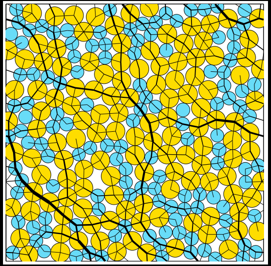

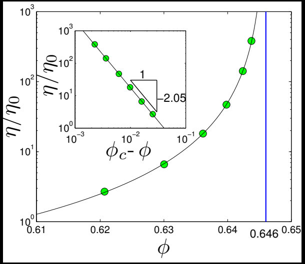

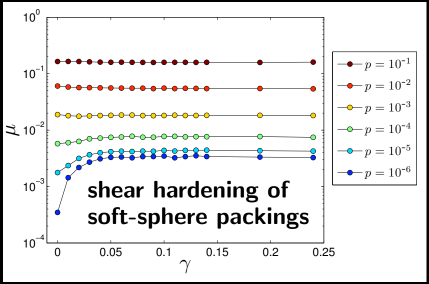

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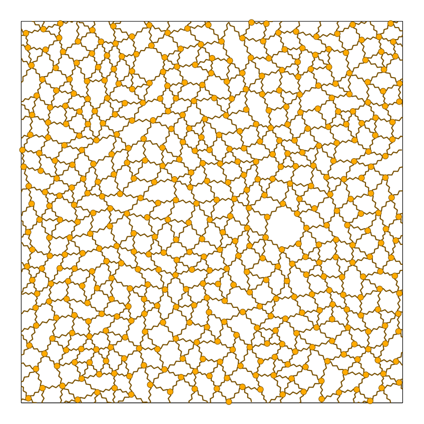

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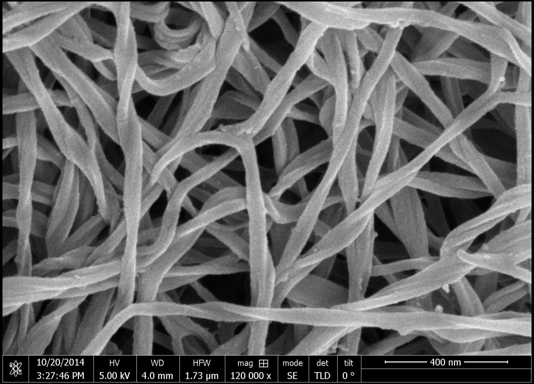

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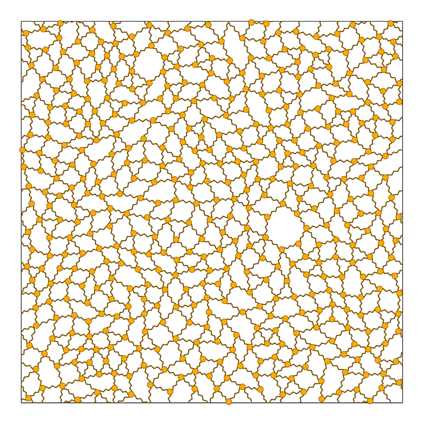
- relation to other jamming problems & some open questions

disordered networks of masses connected by **relaxed** Hookean springs



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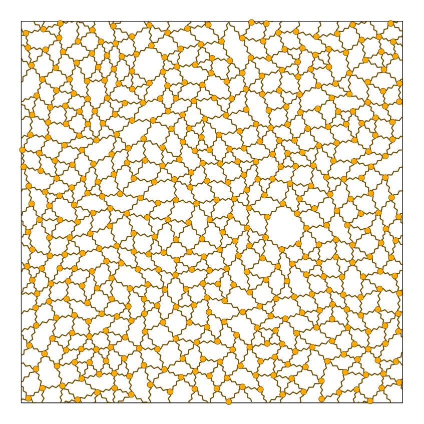
key control parameter: coordination z



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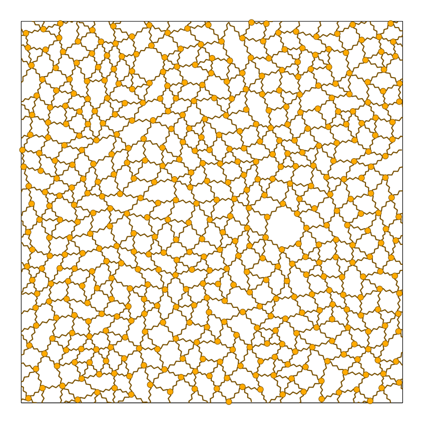

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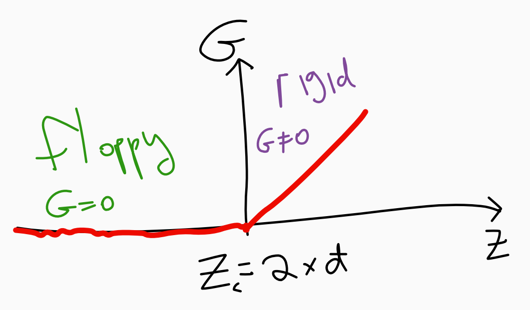

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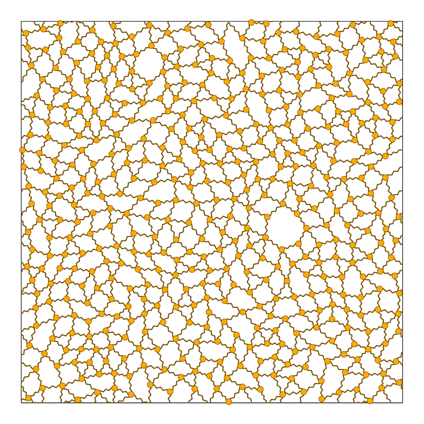
key control parameter: coordination $z < z_{\mathrm{c}} \equiv 2 \times \mathsf{d}$

'floppy' networks



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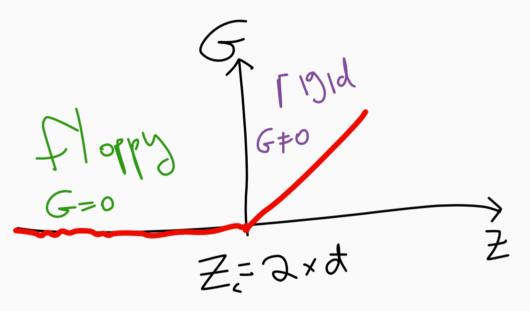
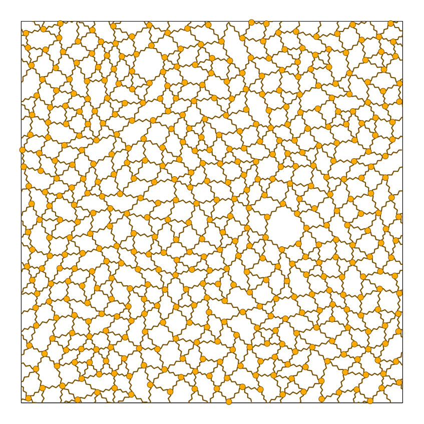


‘floppy’ networks feature ‘floppy modes’



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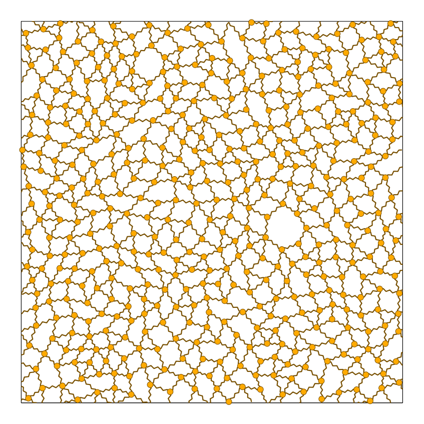

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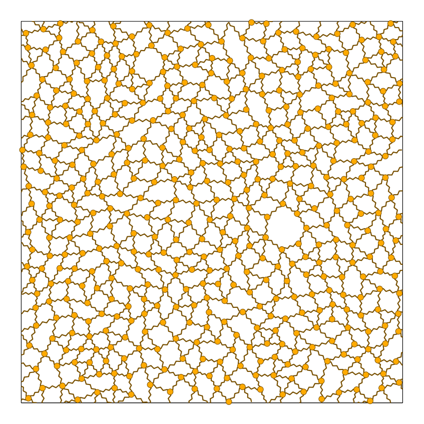

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$$\text{if } \hat{\boldsymbol{n}}_{ij} \cdot (\boldsymbol{u}_j - \boldsymbol{u}_i) = 0 \quad \text{for all springs } i, j$$

$\Rightarrow$ $\boldsymbol{u}$ is a **floppy mode**

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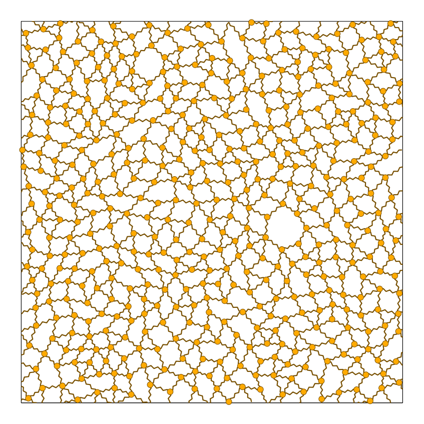

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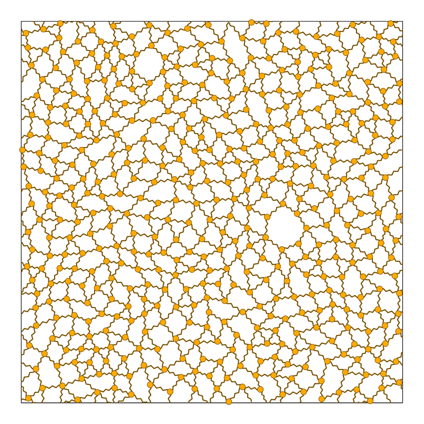

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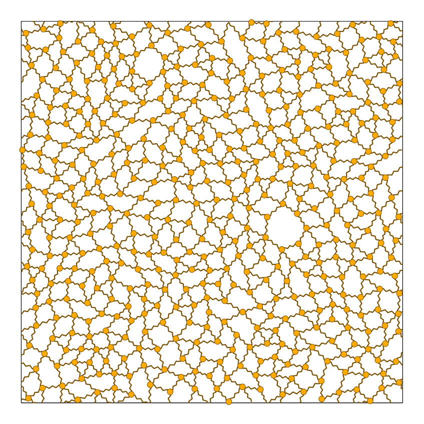
$\mathcal{S}$ is known as the '*compatibility matrix*'

floppy networks **do not** feature ‘states of self-stress’



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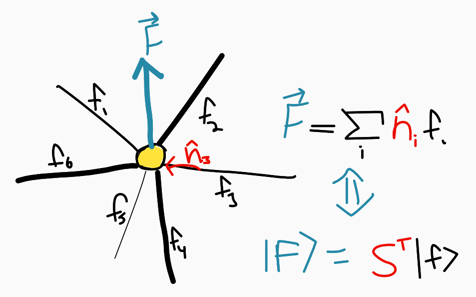

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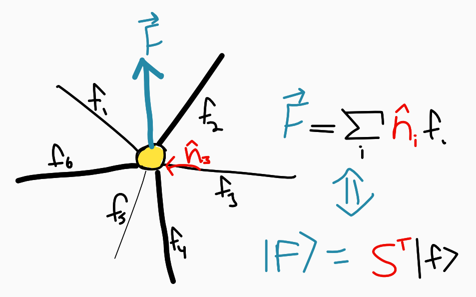

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Wyart (phd thesis, 2005) showed that (for relaxed spring networks)

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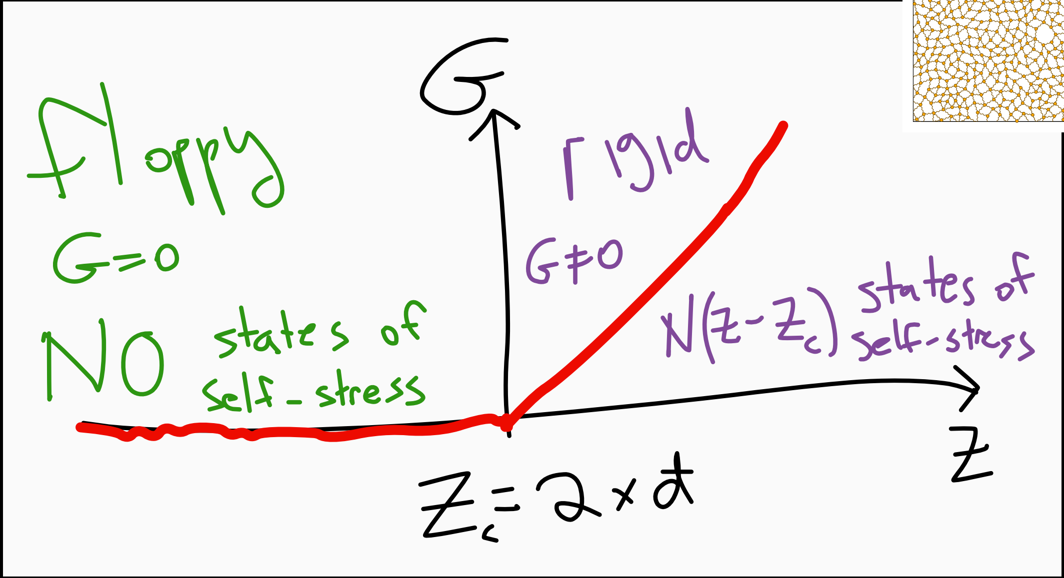
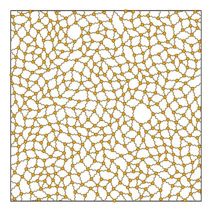
no states-of-self-stress? then $G = 0$.

floppy networks – summary





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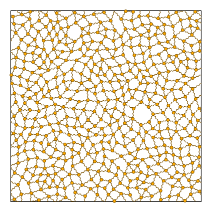
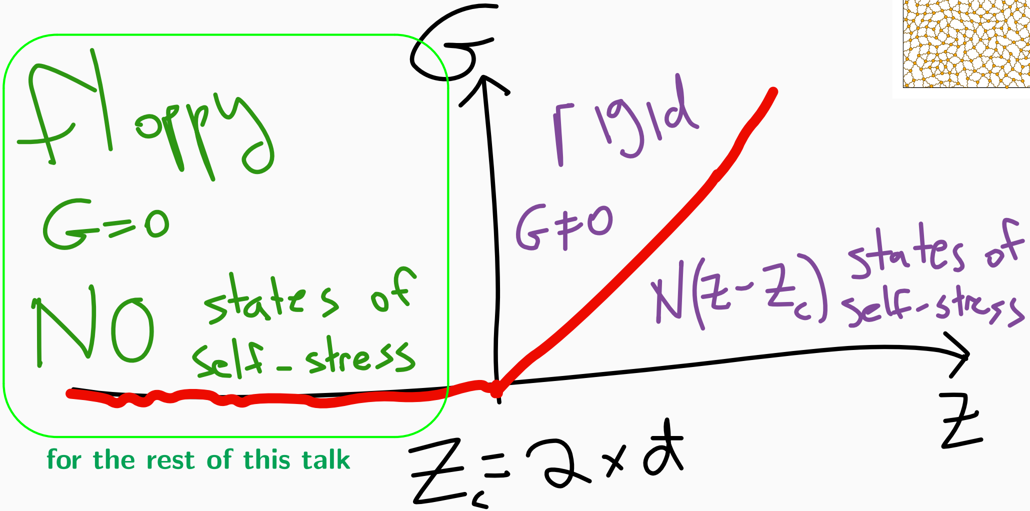
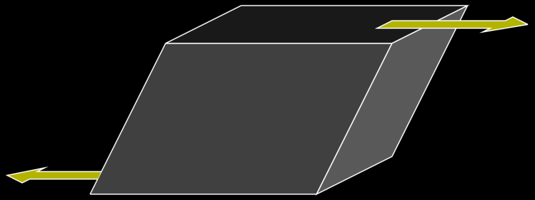




for the rest of this talk

what happens when a floppy network is sheared?



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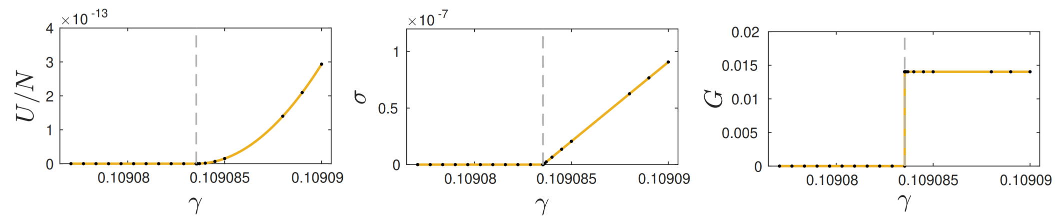

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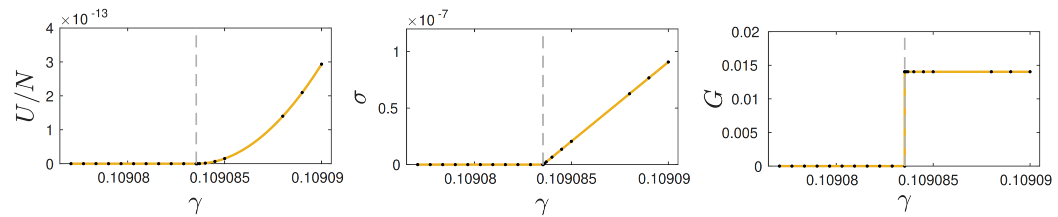

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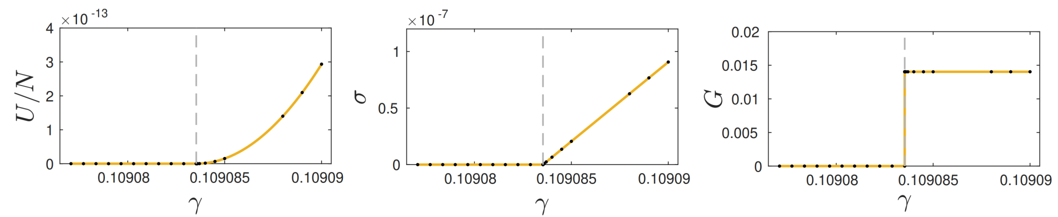

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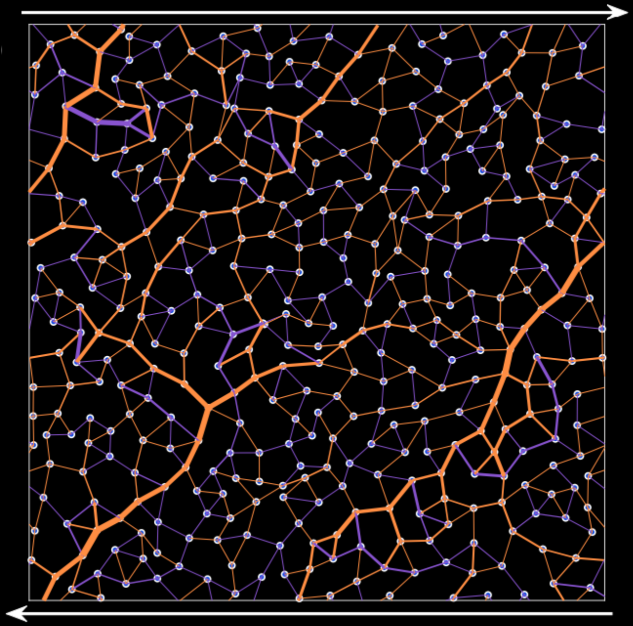
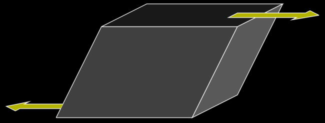
$\Rightarrow$ a **state of self-stress** **developed**

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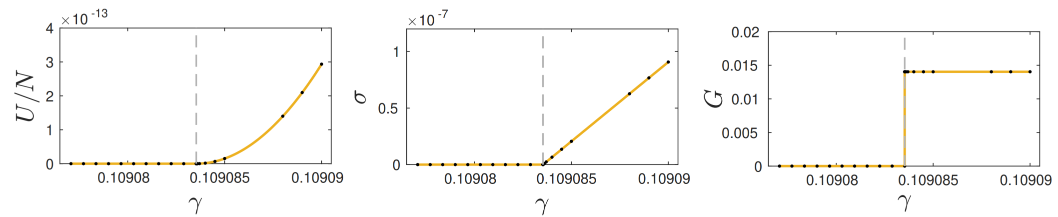

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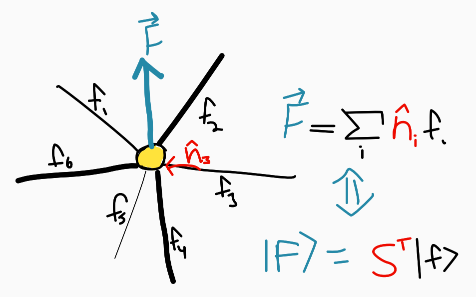
how can this 👆 be quantified?

recall: states of self-stress



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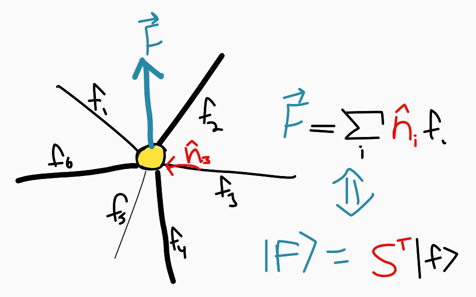
we construct the operator: $\mathcal{S}\mathcal{S}^T$

and consider its **spectrum**



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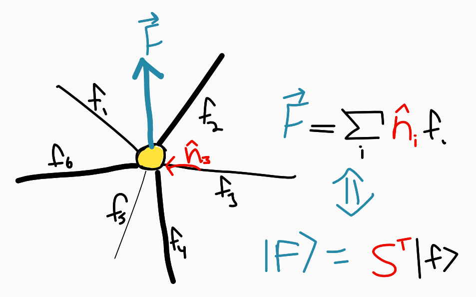

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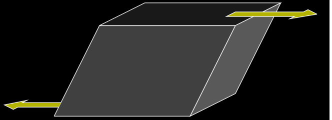
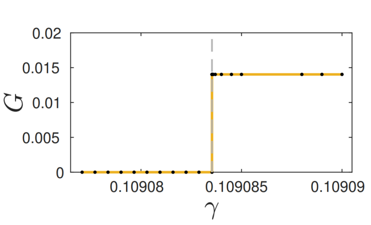
eigenvectors $|f\rangle$: sets of **spring-forces**,

eigenvalues ω^2: **dimensionless force unballance**:

$$\omega^2 = \frac{\langle f|\mathcal{S}\mathcal{S}^T|f\rangle}{\langle f|f\rangle} = \frac{\langle F|F\rangle}{\langle f|f\rangle}$$





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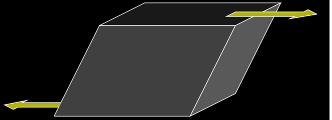
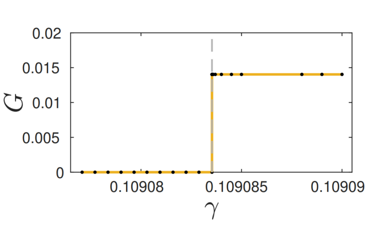
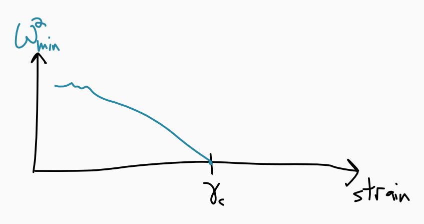
we expect:

at γ_c, $\omega^2 \to 0$







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spectrum of $\mathcal{S}\mathcal{S}^T$ in sheared floppy networks $\left(\mathcal{S}\mathcal{S}^T|f\rangle = \omega^2|f\rangle\right)$

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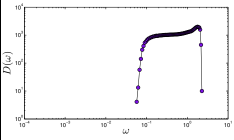

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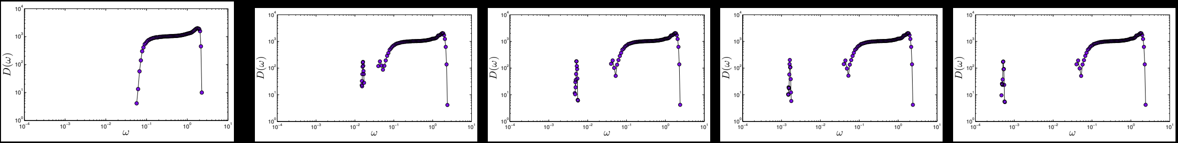

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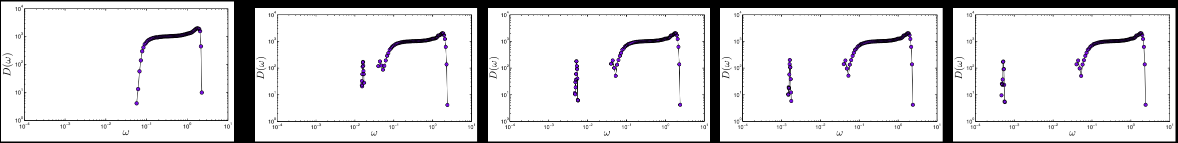

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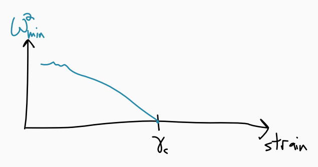
development of a **state of self-stress** $\quad\Leftrightarrow\quad \omega_{\min}^2 \to 0$



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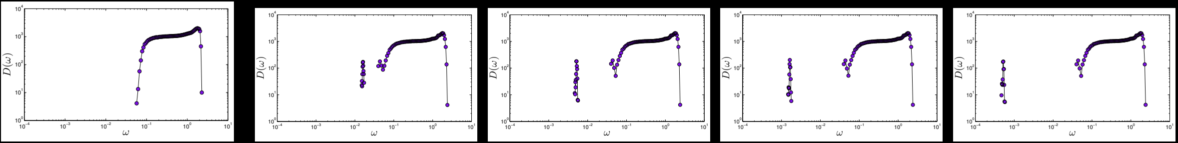

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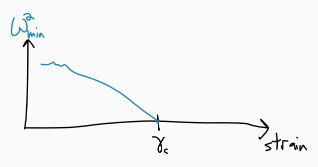
how does ω^2_{min} vanish?



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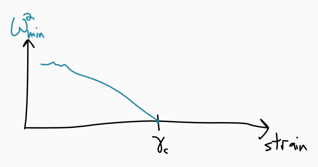

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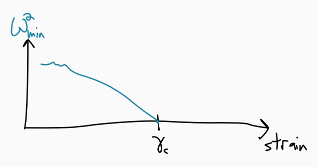

the (quasistatic) dynamics of a **floppy network** under shear is **ill-defined**

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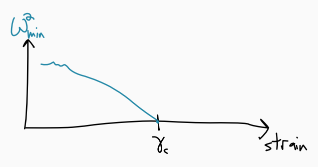

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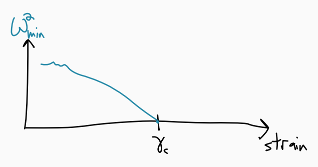

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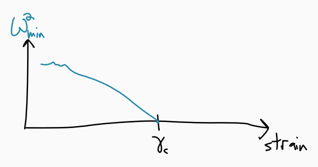

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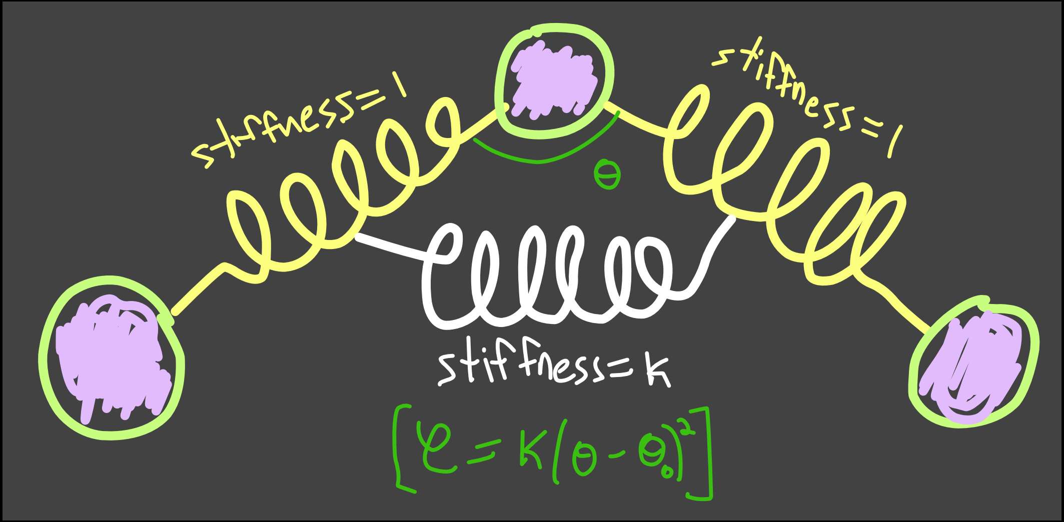
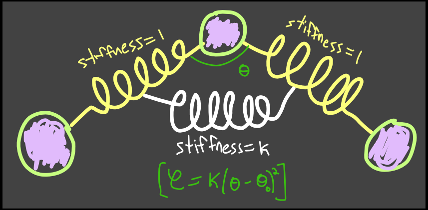
to proceed, we introduce a **weak interaction** of typical stiffness κ, that **eliminates** the indeterminacy of dynamics/mechanics introducing weak interactions

introducing weak interactions



after introducing weak interactions of stiffness κ, **indeterminacy** is **removed**.



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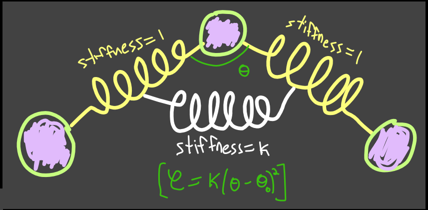

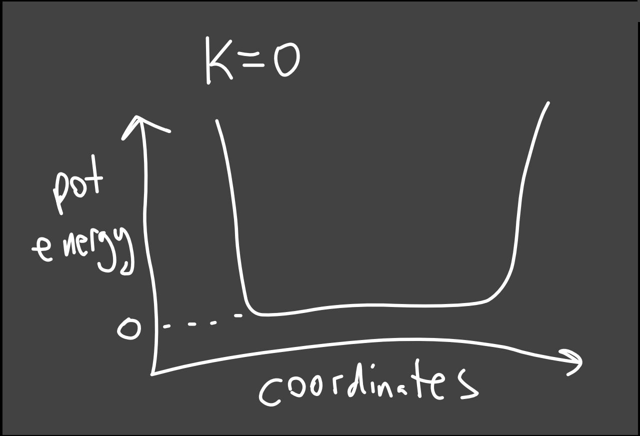

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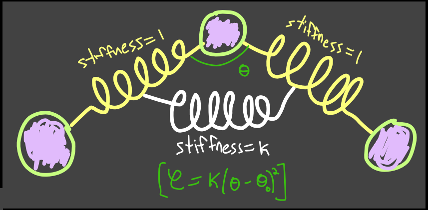

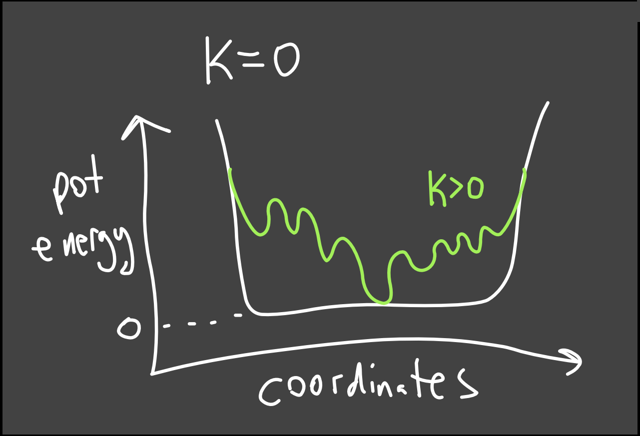

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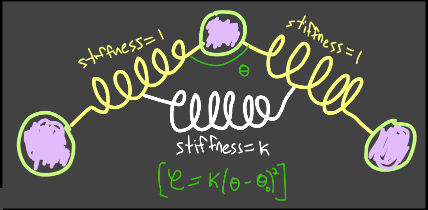

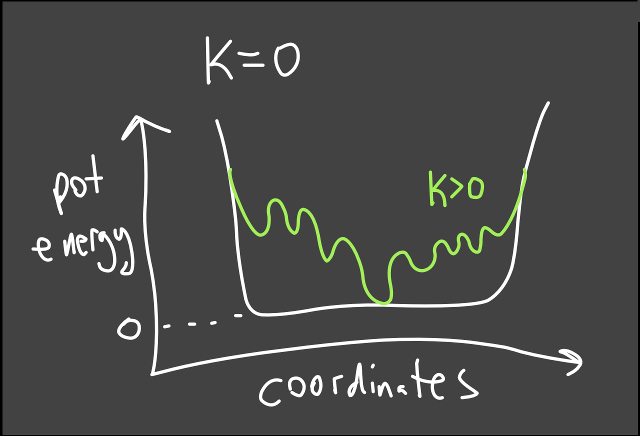

($\kappa > 0$ is a singular perturbation)

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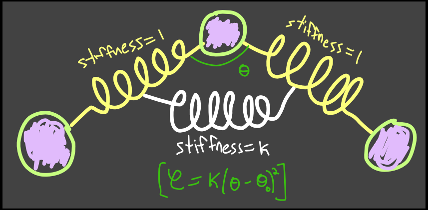

one useful limit is $\boldsymbol{\kappa} \longrightarrow \mathbf{0}^+$, then one finds:

$$\omega^2_{\min} \sim \gamma_c - \gamma$$

(recall $\mathcal{S}\mathcal{S}^T|f\rangle = \omega^2|f\rangle$)

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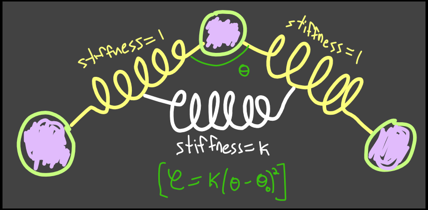

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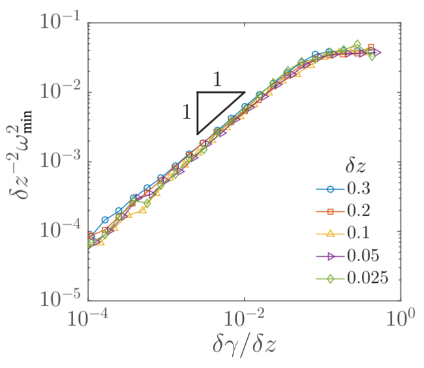

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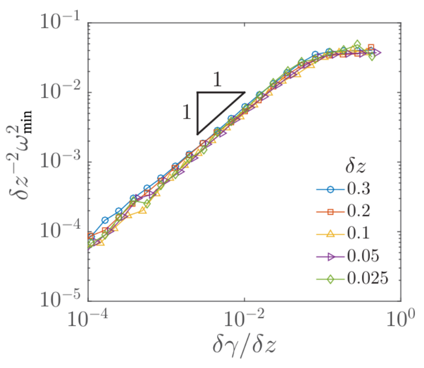
strain stiffening

operator: $\mathcal{S}\mathcal{S}^T$, $\quad \omega_{\mathrm{min}}^2 \sim \gamma_c - \gamma$



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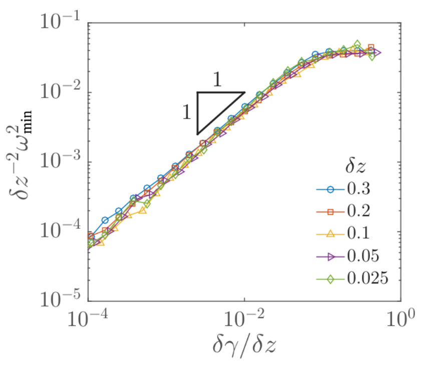

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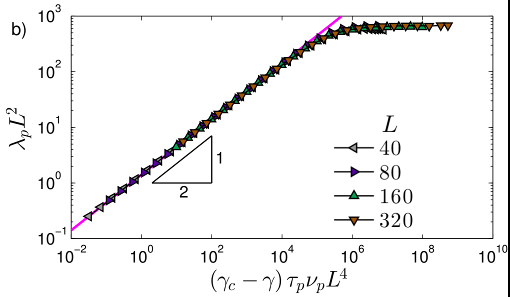
plastic instabilities in elastic solids

operator: $\boldsymbol{\mathcal{H}} = \frac{\partial^2 U}{\partial \boldsymbol{x} \partial \boldsymbol{x}}$, $\omega_{\min}^2 \sim \sqrt{\gamma_c - \gamma}$



EL, PRE 2016

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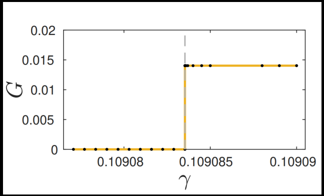

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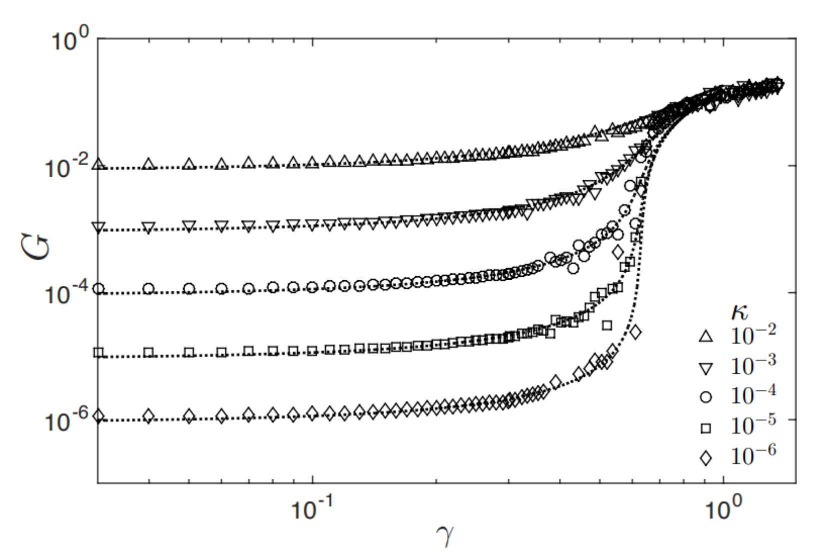

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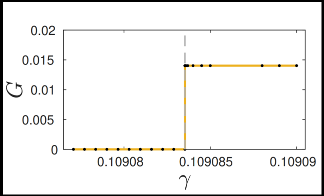

R. Rens et al. J. Phys. Chem. B, 2016

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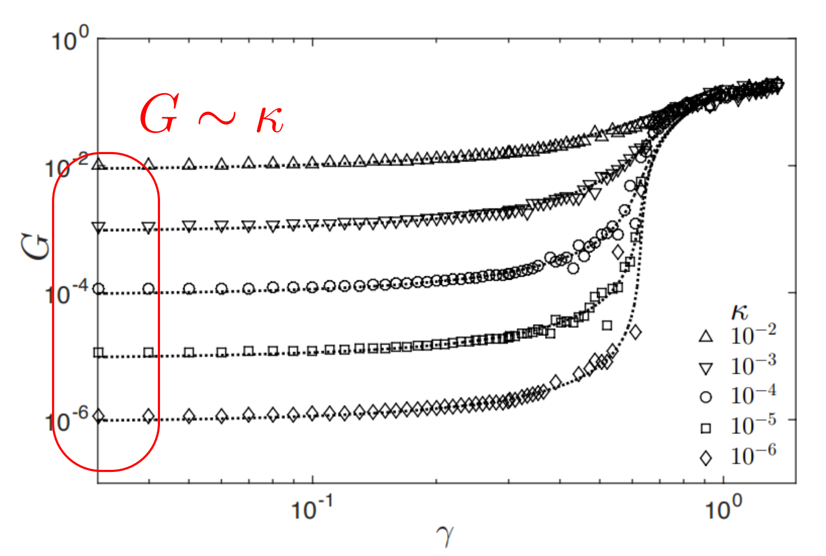

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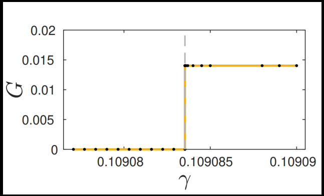

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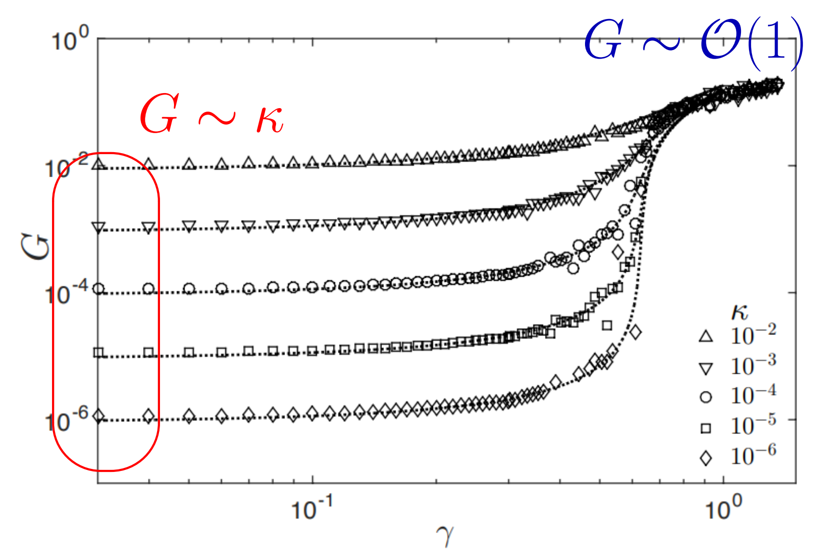

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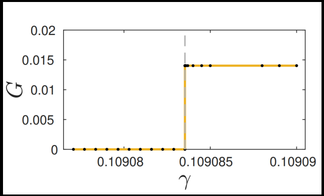

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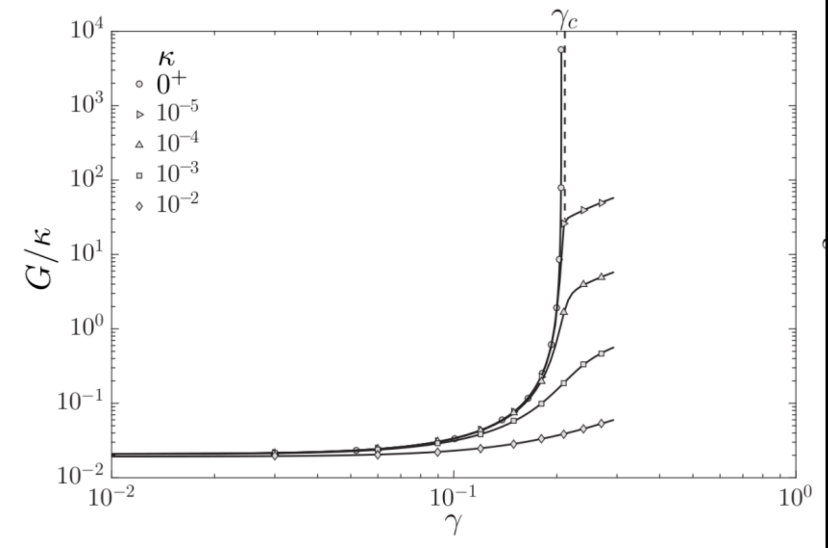

R. Rens, C. Villarroel, G. Düring, and EL, PRE 2018

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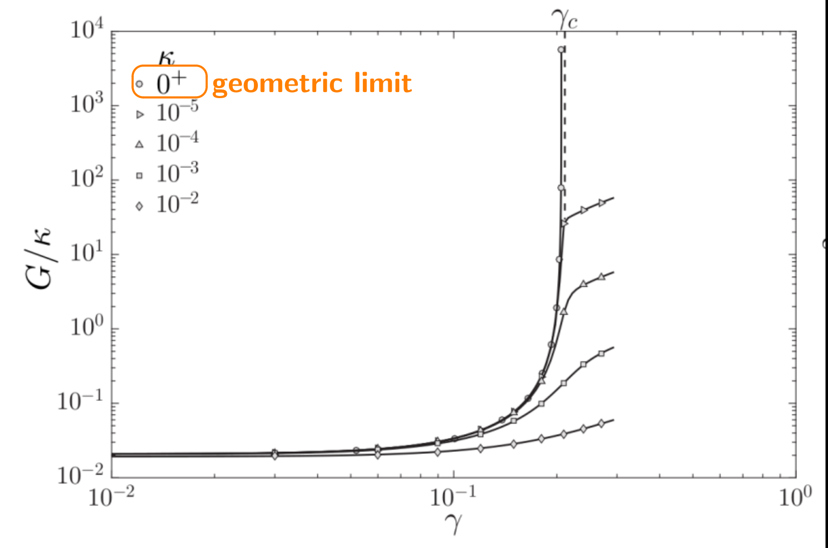

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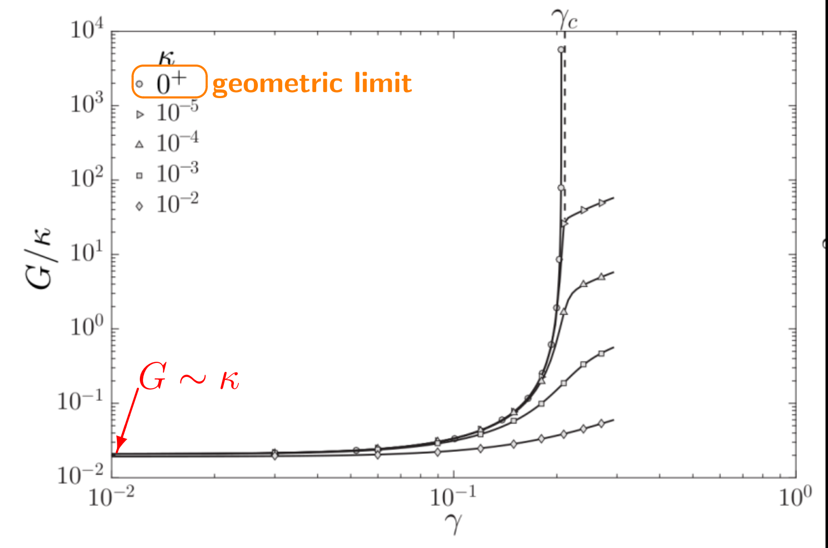

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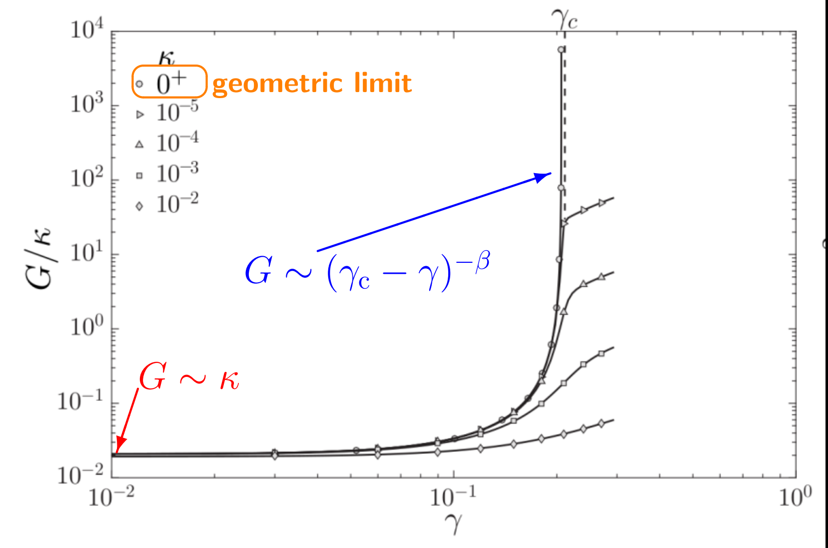

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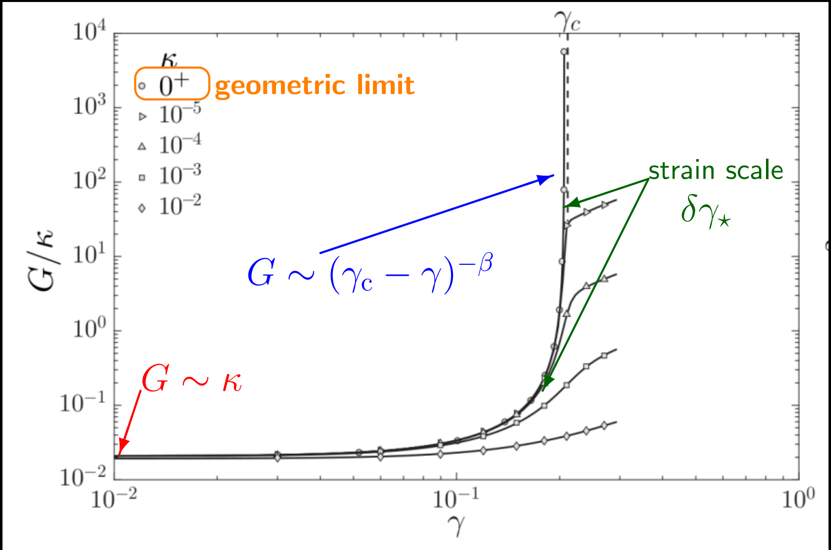

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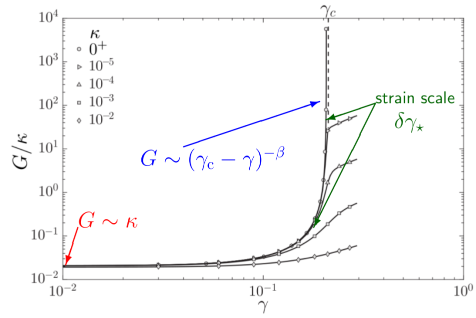
- in **isotropic** states $G \sim \kappa$
- if $\delta\gamma > \delta\gamma_\star(\kappa)$, $G \sim (\gamma_c - \gamma)^{-\beta}$



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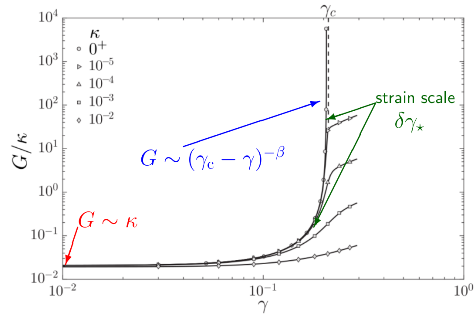
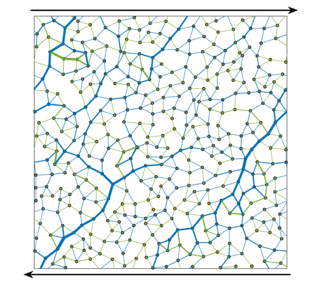


how can these observations be understood?

consider a shear-stiffened network with $\kappa = 0$;



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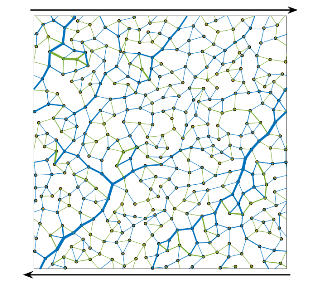
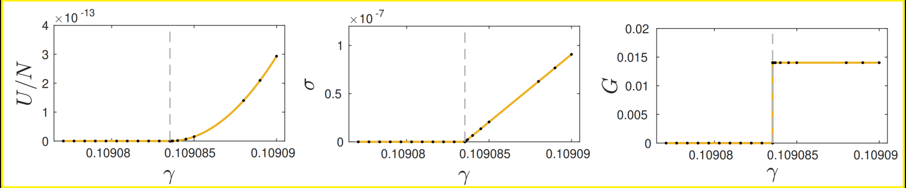
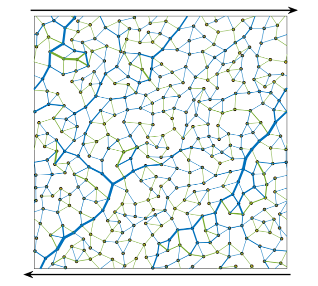


recall:



consider a shear-stiffened network with $\kappa = 0$;

counting DOF vs. interactions, $\sim N$ floppy modes exist



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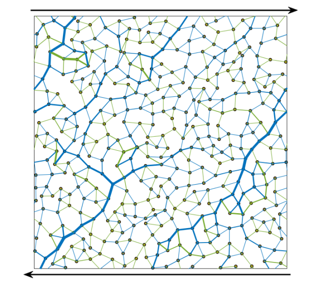

1) expand the energy in the floppy-mode space:

$$U(u) \simeq \cancel{\frac{1}{2}\frac{\partial^2 U}{\partial x^2}u^2} + \cancel{\frac{1}{6}\frac{\partial^3 U}{\partial x^3}u^3} + \frac{1}{24}\frac{\partial^4 U}{\partial x^4}u^4$$

floppy modes stability

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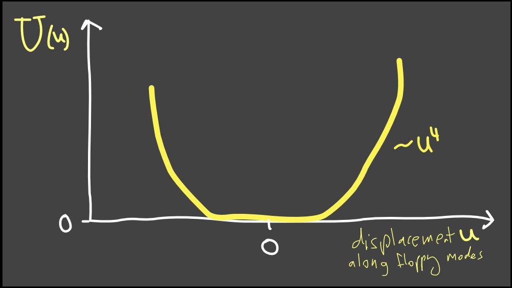

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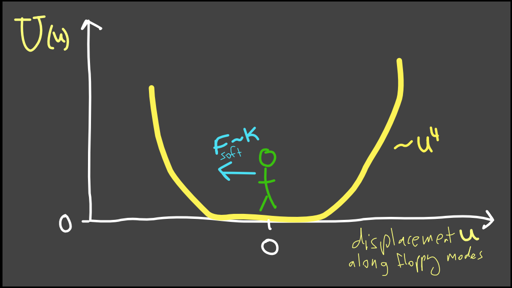

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floppy modes stability

2) turn on weak interactions. Nodes are now unballanced by a force $F_{\text{soft}} \sim \kappa$

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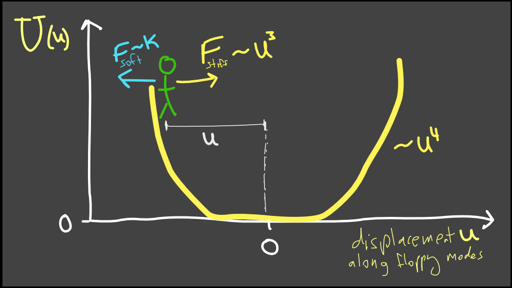

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when **anharmonic** force balances weak force: $F_{\text{soft}} \sim \kappa \sim F_{\text{stiff}} \sim u_\star^3$

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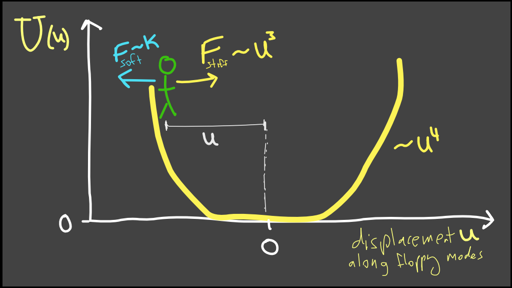

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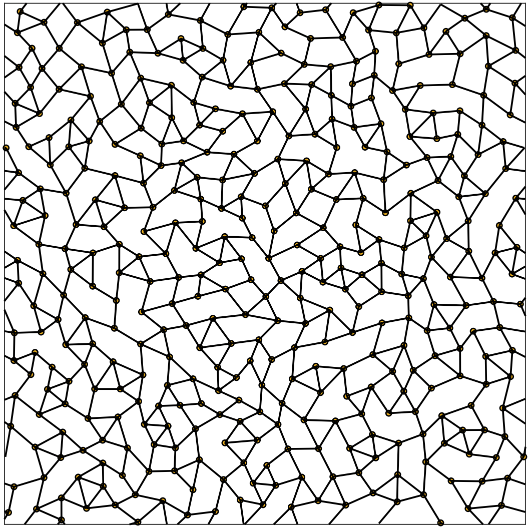
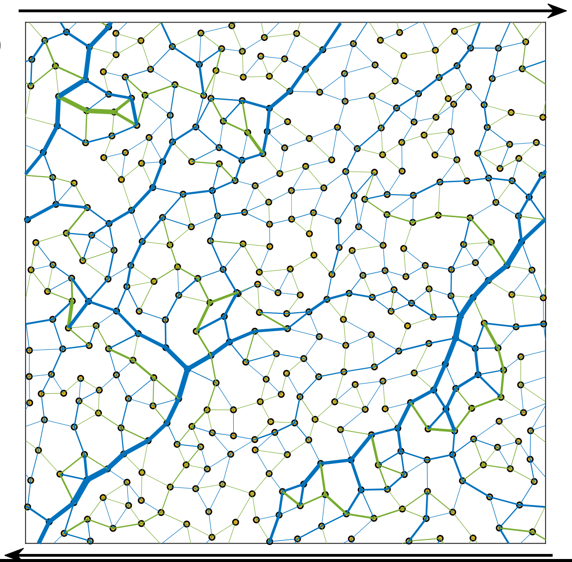
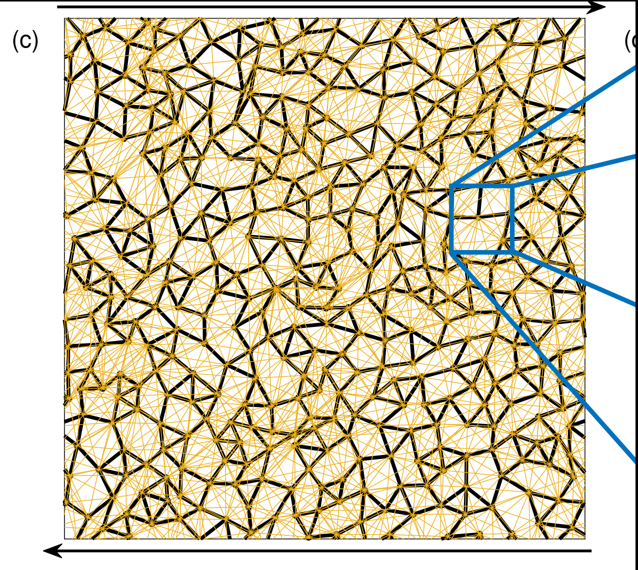
$$\boxed{u_\star \sim \kappa^{1/3}}$$

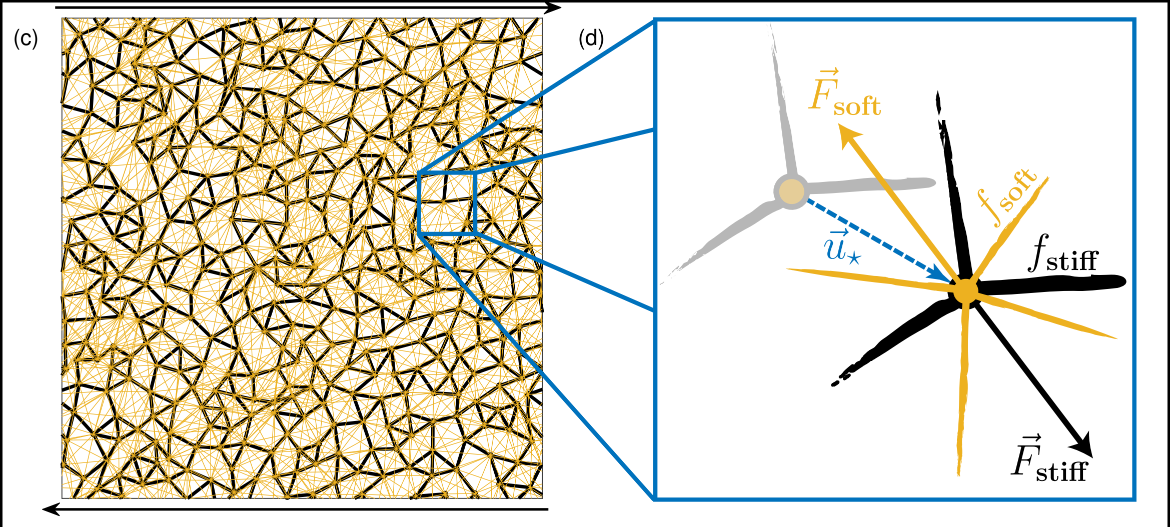
(a)

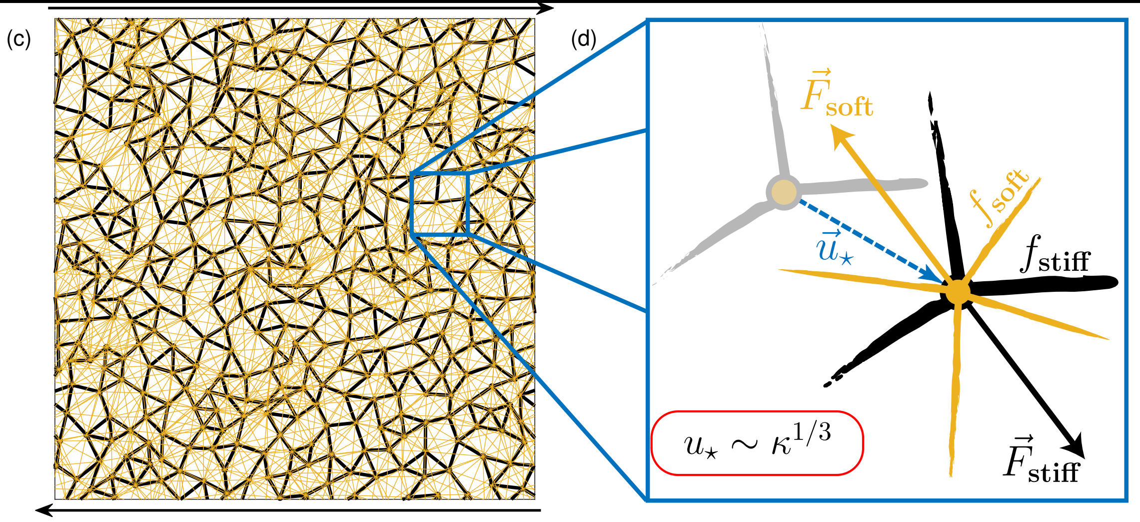


(b)



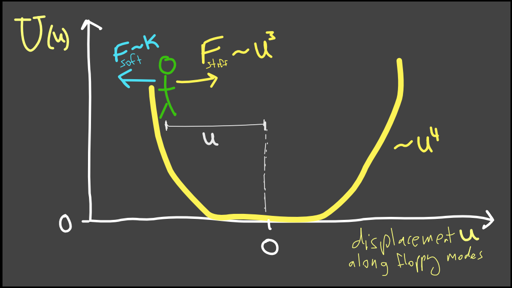

(c)


(c)
(d)
$\vec{F}_{\text{soft}}$
f_{soft}
f_{stiff}
$\vec{u}_{\star}$
$\vec{F}_{\text{stiff}}$



(c)
(d)
$\vec{F}_{\rm soft}$
$f_{\rm soft}$
$f_{\rm stiff}$
$\vec{u}_\star$
$u_\star \sim \kappa^{1/3}$
$\vec{F}_{\rm stiff}$

why do we need this 2-step perturbation approach?



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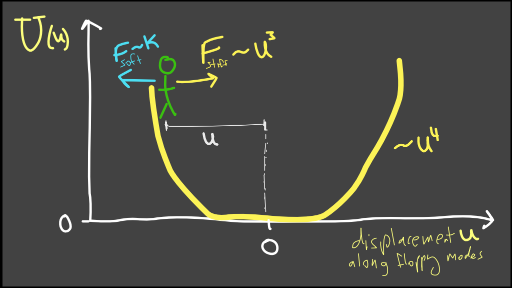

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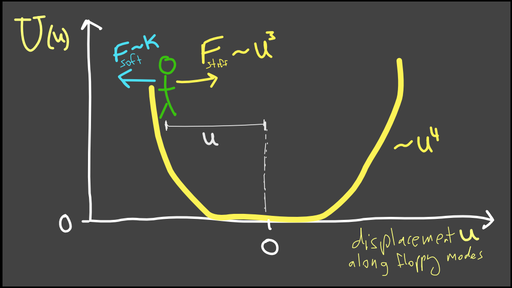
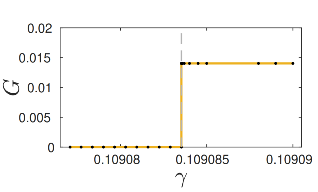
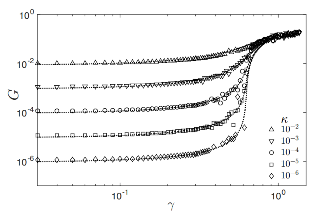


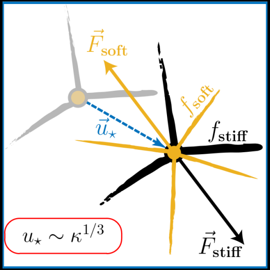
1) theoretical handle (to be explained hereafter)

2) allows to *simulate* systems **at the critical strain** (unfeasible otherwise)



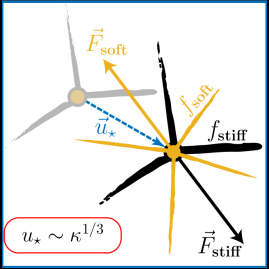


properties of perturbed ($\kappa > 0$), strain-stiffened states



properties of perturbed ($\kappa > 0$), strain-stiffened states

1) displacements $u_\star$ distort (and ruin) the $\kappa = 0$ **state-of-self-stress**



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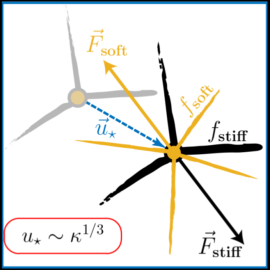

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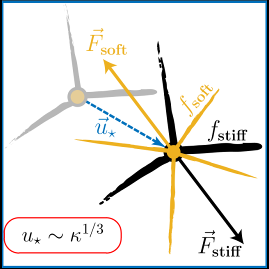

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since the stiff network needs to balance the soft ($\sim \kappa$) force, one expects an **amplification** over $\sim \kappa$:

shear stress $\quad \sigma \sim \kappa \Big/ \sqrt{\frac{\langle f|\mathcal{S}\mathcal{S}^T|f\rangle}{\langle f|f\rangle}} \sim \kappa^{2/3}$

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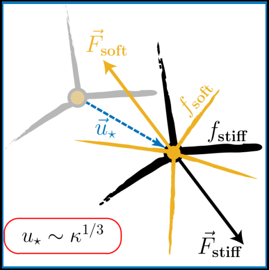
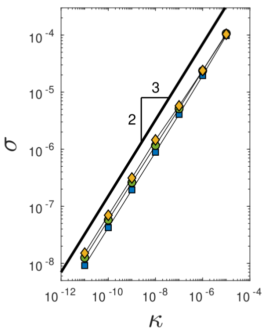
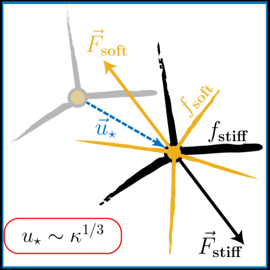




properties of perturbed ($\kappa > 0$), strain-stiffened states



2) displacements $u_\star$ distort (and ruin) the $\kappa = 0$ **zero modes**:

properties of perturbed ($\kappa > 0$), strain-stiffened states

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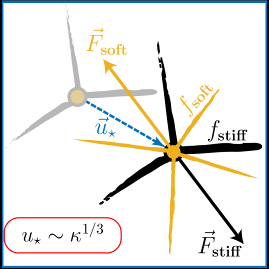
$$\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}_1 + \boldsymbol{\mathcal{H}}_2 + \boldsymbol{\mathcal{H}}_{\rm soft}$$



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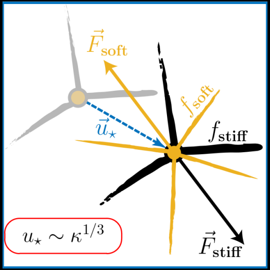

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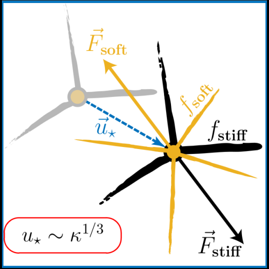
stiffness

force

bending

properties of perturbed ($\kappa > 0$), strain-stiffened states



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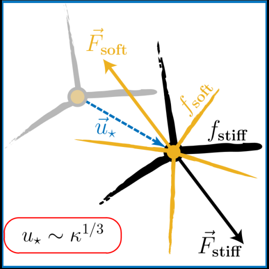
stiffness　force　bending

stiffness term:

$$\boldsymbol{\mathcal{H}}_1 = \sum\nolimits_{\langle i,j \rangle} \boldsymbol{n}_{ij} \bigotimes \boldsymbol{n}_{ij} \quad \Rightarrow \quad \delta \boldsymbol{\mathcal{H}}_1 \sim \delta \boldsymbol{n} \bigotimes \delta \boldsymbol{n} \sim u_\star^2 \sim \kappa^{2/3}$$

properties of perturbed ($\kappa > 0$), strain-stiffened states



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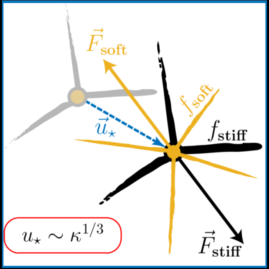

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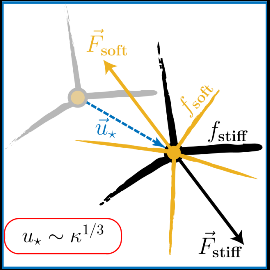
force term:

$$\mathcal{H}_2 \sim f \quad \Rightarrow \quad \delta\mathcal{H}_2 \sim f \sim \kappa \Big/ \sqrt{\frac{\langle f|SS^T|f\rangle}{\langle f|f\rangle}} \sim \kappa^{2/3}$$

properties of perturbed ($\kappa > 0$), strain-stiffened states



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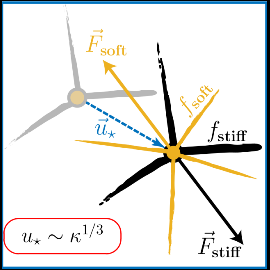
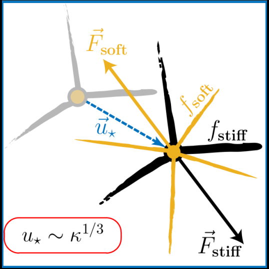


$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_{\mathrm{soft}}$$

stiffness $\sim \kappa^{2/3}$ force $\sim \kappa^{2/3}$ bending $\sim \kappa$

properties of perturbed ($\kappa > 0$), strain-stiffened states



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stiffness $\sim \kappa^{2/3}$

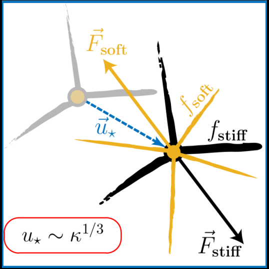
force $\sim \kappa^{2/3}$

bending $\sim \kappa$

new frequency of previously-zero-modes $\omega(\kappa) \sim \kappa^{1/3}$

properties of perturbed ($\kappa > 0$), strain-stiffened states

3) shear modulus $G = G_{\mathrm{affine}} + G_{\mathrm{nonaffine}}$



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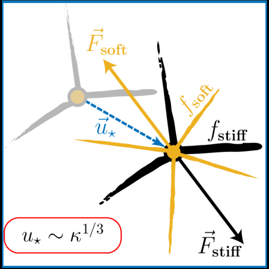

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properties of perturbed ($\kappa > 0$), strain-stiffened states

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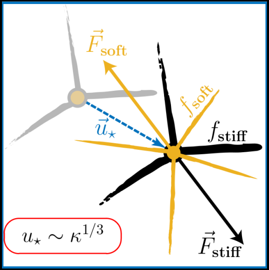

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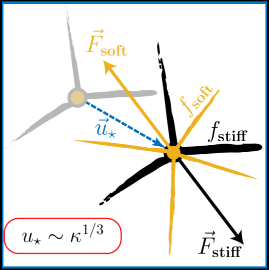
regular

??

why worry?



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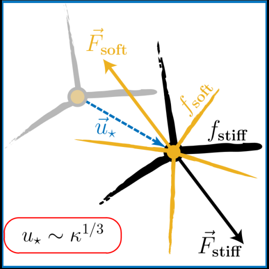

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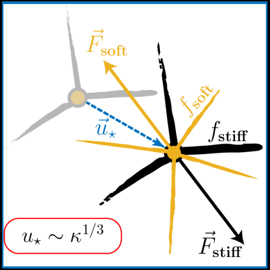
$$G_{\mathrm{nonaffine}} = \sum_\ell \frac{\left(\boldsymbol{F}_\gamma \cdot \boldsymbol{\psi}_\ell\right)^2}{\omega_\ell^2}$$

properties of perturbed ($\kappa > 0$), strain-stiffened states



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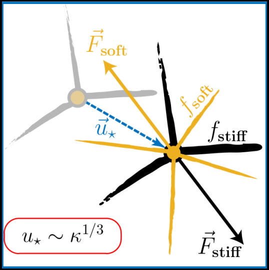
$$G_{\text{nonaffine}} = \sum_\ell \frac{(\boldsymbol{F}_\gamma \cdot \boldsymbol{\psi}_\ell)^2}{\omega_\ell^2}$$

$$\omega_{\text{soft}} \sim \kappa^{1/3}$$

properties of perturbed ($\kappa > 0$), strain-stiffened states



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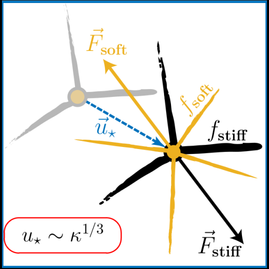
$$G_{\text{nonaffine}} = \sum_\ell \frac{\left(\boldsymbol{F}_\gamma \cdot \boldsymbol{\psi}_\ell\right)^2}{\omega_\ell^2}$$

$\omega_{\text{soft}} \sim \kappa^{1/3}$ but $\boldsymbol{F}_\gamma \equiv \dfrac{\partial^2 U}{\partial\gamma\partial\boldsymbol{x}} \simeq \mathcal{S}^T|\partial r/\partial\gamma\rangle$ (in the $\kappa \to 0$ limit)

and $\mathcal{S}|\boldsymbol{\psi}\rangle \sim \omega \sim \kappa^{1/3}$

properties of perturbed ($\kappa > 0$), strain-stiffened states



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$$G_{\rm nonaffine} = \sum_\ell \frac{(\boldsymbol{F}_\gamma \cdot \boldsymbol{\psi}_\ell)^2}{\omega_\ell^2} \sim \frac{\kappa^{2/3}}{\kappa^{2/3}} \sim \kappa^0$$ is regular too!

$\omega_{\rm soft} \sim \kappa^{1/3}$

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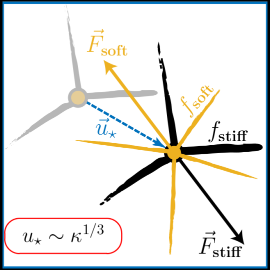
properties of perturbed ($\kappa > 0$), strain-stiffened states



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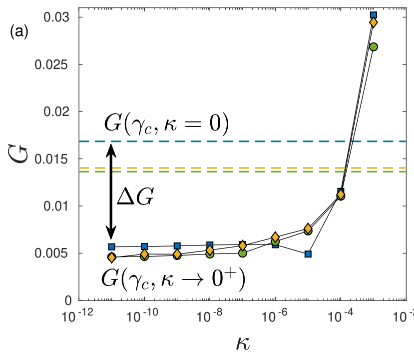

properties of perturbed ($\kappa > 0$), strain-stiffened states

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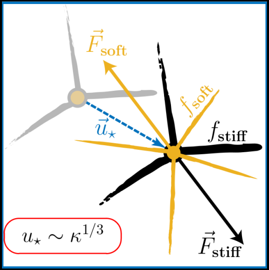
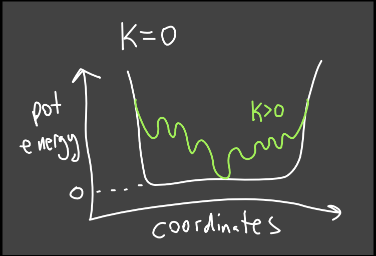
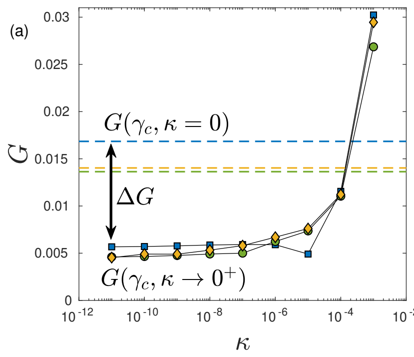
again, $\kappa > 0$ is a **singular perturbation**



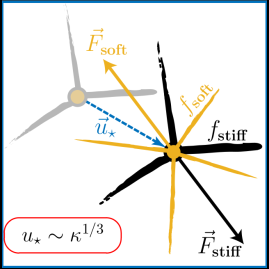




properties of perturbed ($\kappa > 0$), strain-stiffened states

4) nonlinear shear modulus $dG/d\gamma$

$$\frac{dG}{d\gamma} \simeq \frac{1}{V}\sum_{\ell m n} \frac{(\boldsymbol{\psi}_\ell \cdot \boldsymbol{F}_\gamma)(\boldsymbol{\psi}_m \cdot \boldsymbol{F}_\gamma)(\boldsymbol{\psi}_n \cdot \boldsymbol{F}_\gamma)\,(\boldsymbol{\mathcal{U}}''' \,\vdots\, \boldsymbol{\psi}_\ell \boldsymbol{\psi}_m \boldsymbol{\psi}_n)}{\omega_\ell^2 \omega_m^2 \omega_n^2} + \mathcal{O}(\boldsymbol{\mathcal{H}}^{-2})$$



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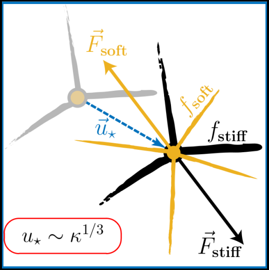

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for soft modes ψ: (i) $\boldsymbol{\psi} \cdot \boldsymbol{F}_\gamma \sim \omega_{\rm soft} \sim \kappa^{1/3}$

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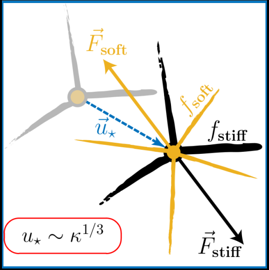
properties of perturbed ($\kappa > 0$), strain-stiffened states



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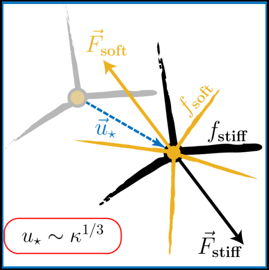
properties of perturbed ($\kappa > 0$), strain-stiffened states



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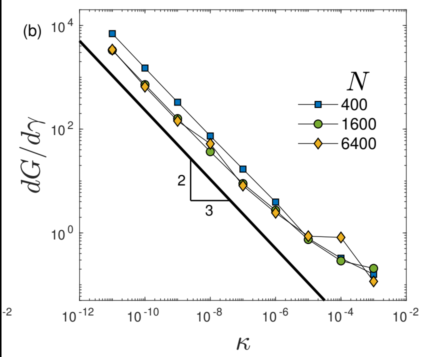

properties of perturbed ($\kappa > 0$), strain-stiffened states

5) nonaffine displacements $\vec{u}_{\text{na}}$

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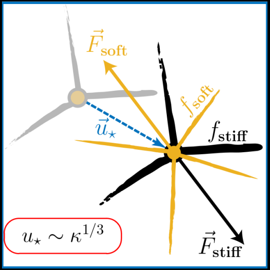

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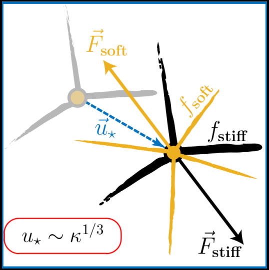

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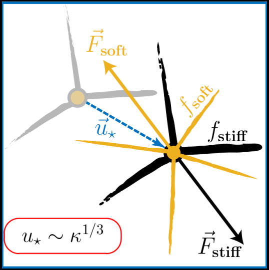
properties of perturbed ($\kappa > 0$), strain-stiffened states



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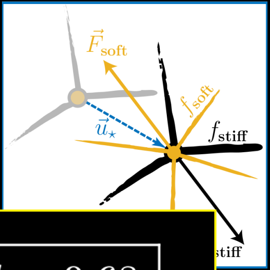
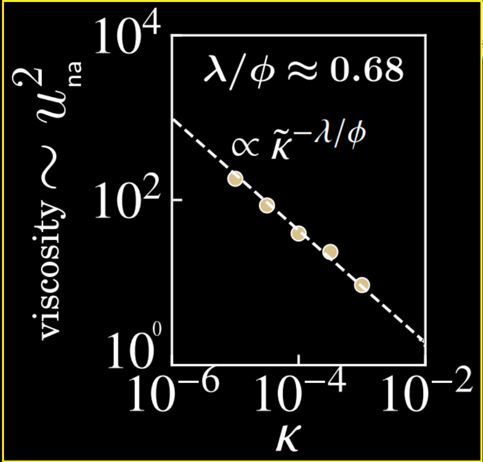
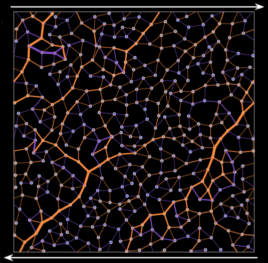




Shivers, Sharma, MacKintosh, arXiv:2203.04891

properties of perturbed ($\kappa > 0$), strain-stiffened states – summary



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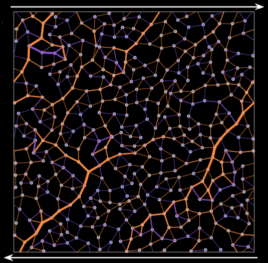

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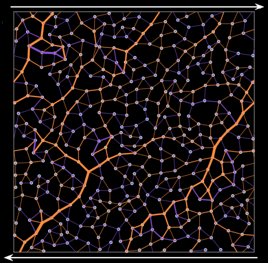

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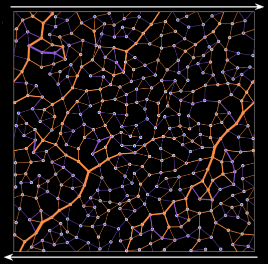

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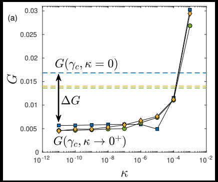

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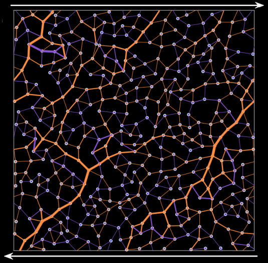

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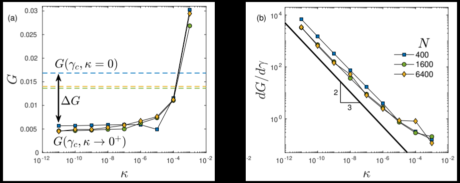

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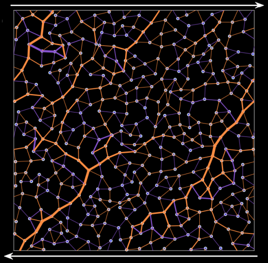

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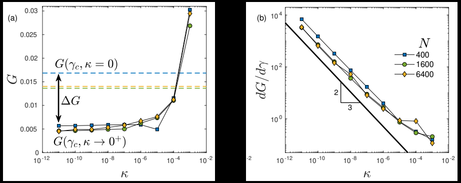

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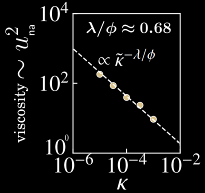
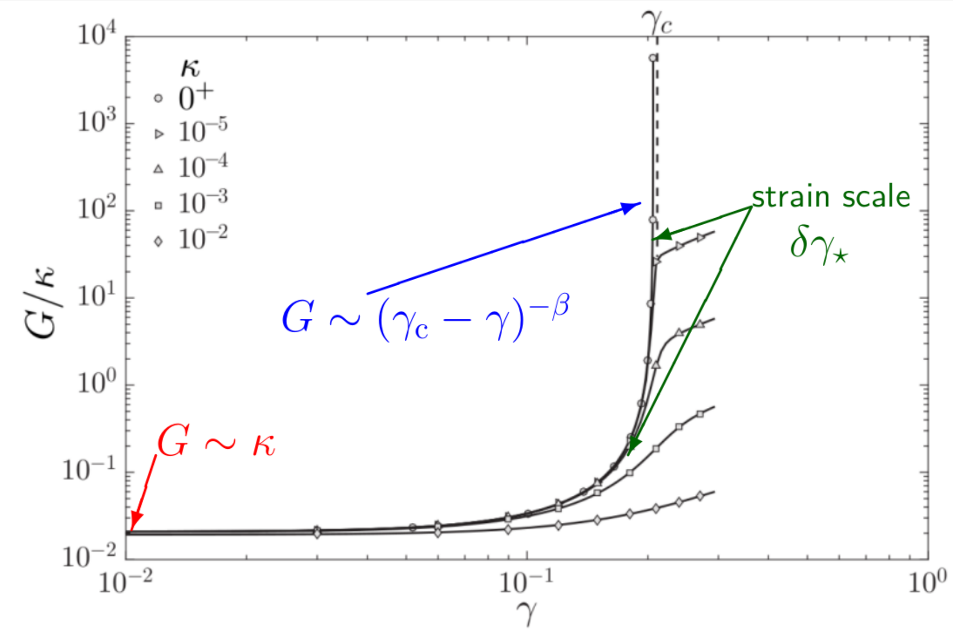


(v) nonaffine displacements $u_{\text{n.a.}} \sim\sim \kappa^{-2/3}$

recap:


γ_c
10^4
10^3
10^2
10^1
10^0
10^{-1}
10^{-2}
G/κ
κ
0^+
10^{-5}
10^{-4}
10^{-3}
10^{-2}
$G \sim (\gamma_c - \gamma)^{-\beta}$
strain scale
$\delta\gamma_\star$
$G \sim \kappa$
10^{-2}
10^{-1}
10^0
γ

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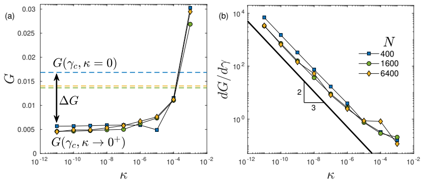

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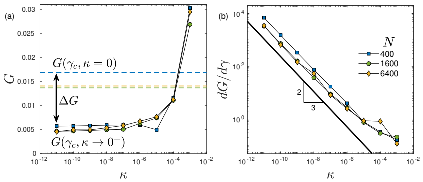

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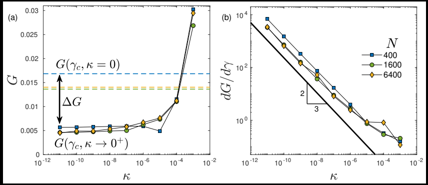

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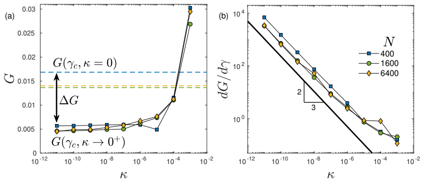

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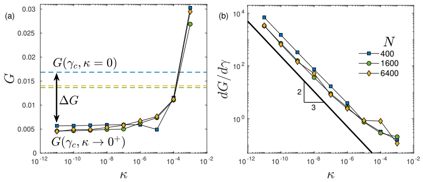

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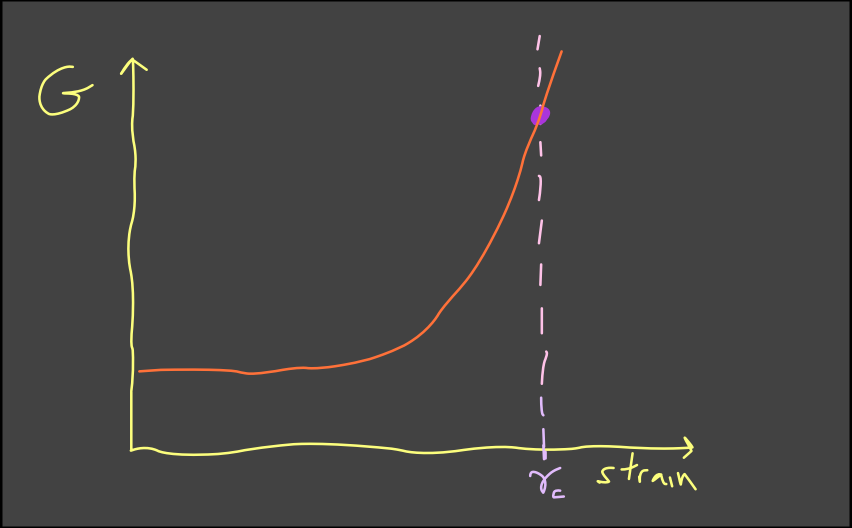

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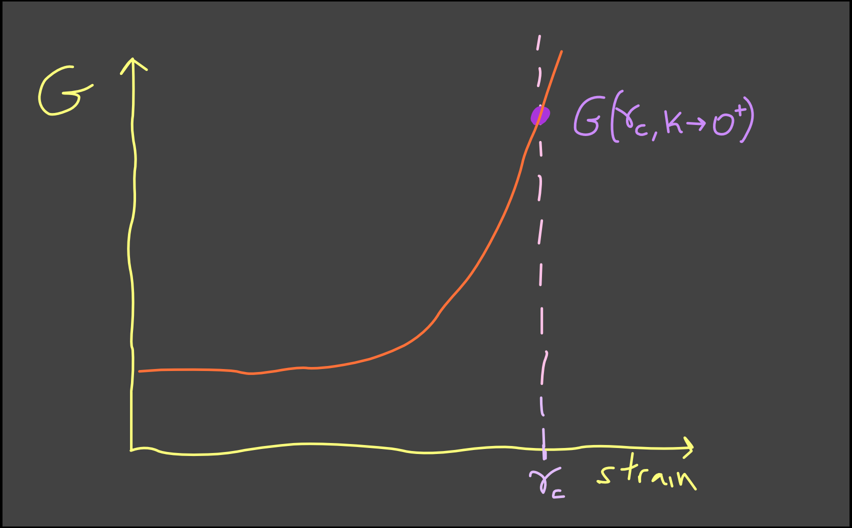

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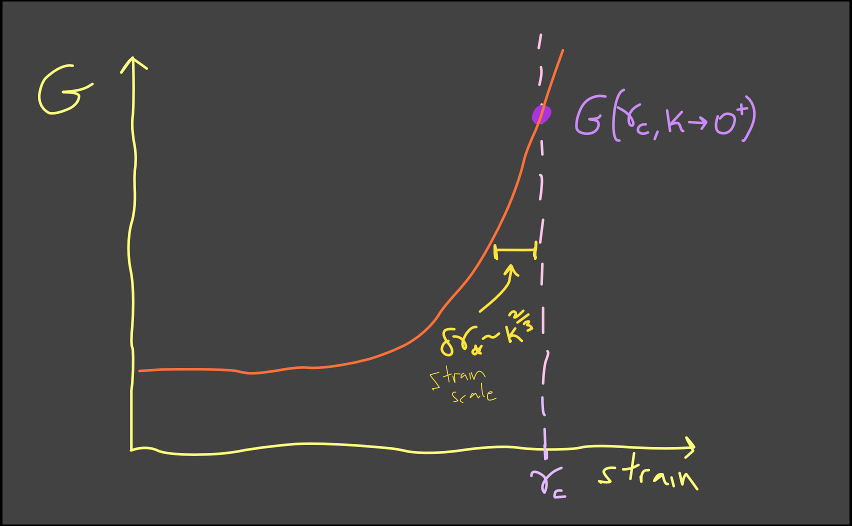

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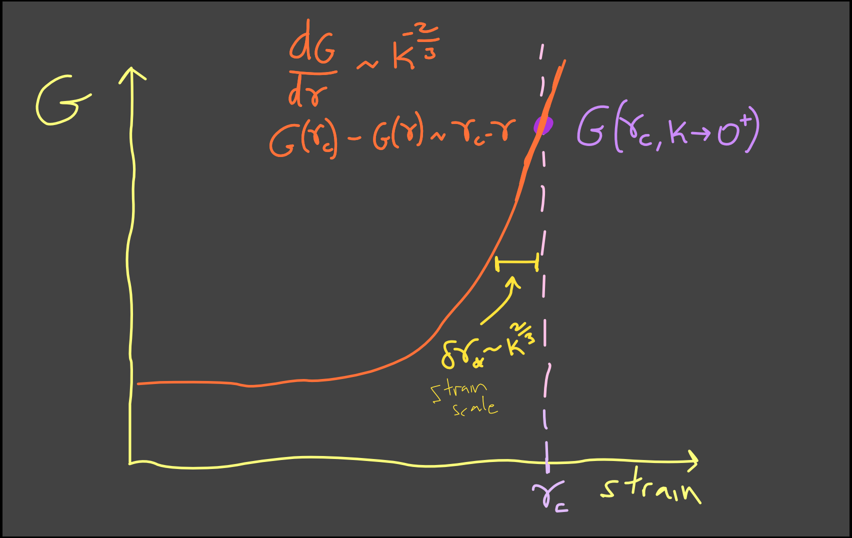

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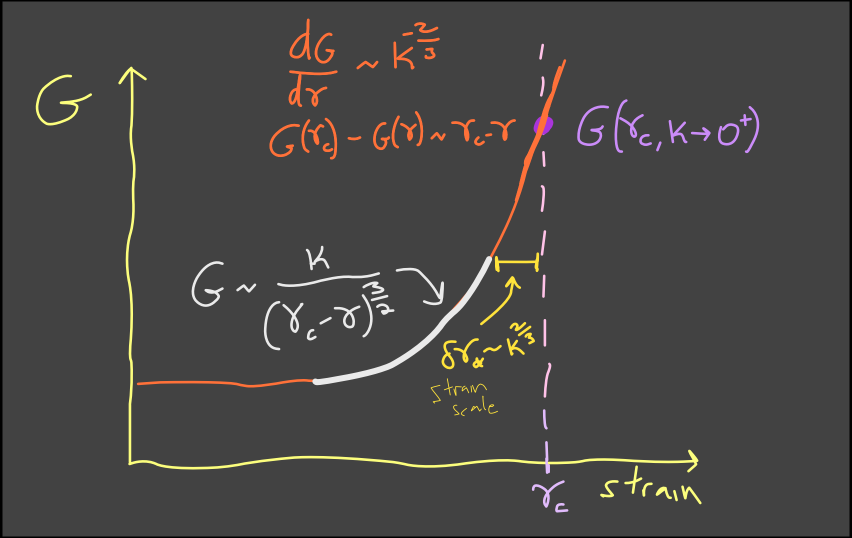
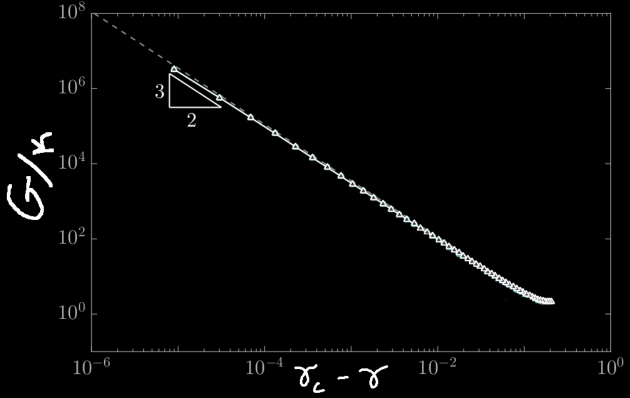
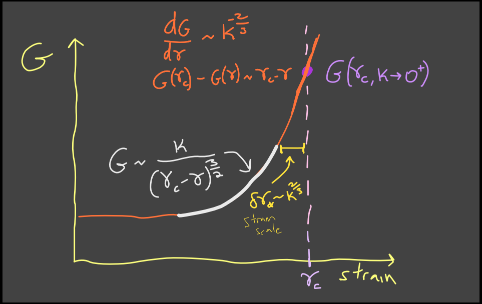


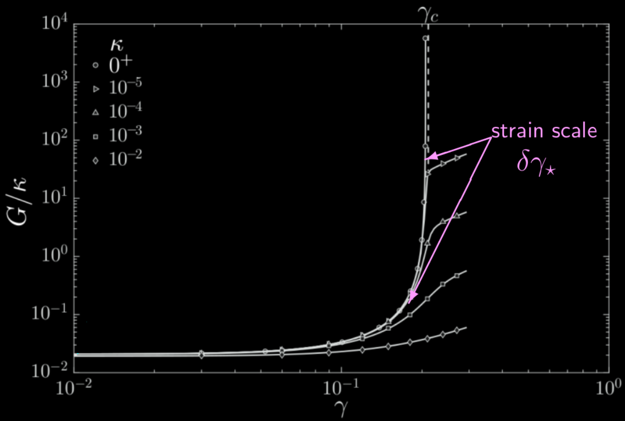
predictions from scaling theory – scaling away from the critical point

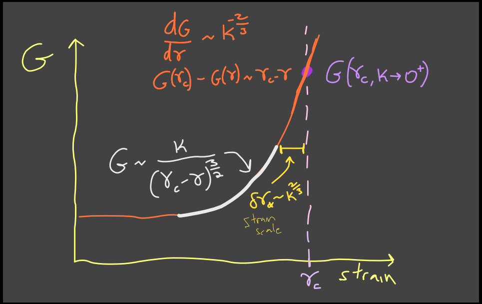




Robbie Rens et al., PRE 2018

predictions from scaling theory – strain scale





Robbie Rens et al., PRE 2018

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→ predictions: (i) **linear** variation of $G \sim \gamma_c - \gamma$ with strain below the critical strain γ_c

(ii) strain scale $\delta\gamma_\star \sim \kappa^{2/3}$

(iii) scaling away from the critical strain $G \sim \dfrac{\kappa}{(\gamma_c - \gamma)^{3/2}}$

open questions

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(i) we expect a diverging correlation length $\xi(\kappa) \sim \dfrac{1}{\sqrt{\frac{\langle f|\mathcal{S}\mathcal{S}^T|f\rangle}{\langle f|f\rangle}}} \sim \dfrac{1}{\kappa^{1/3}}$

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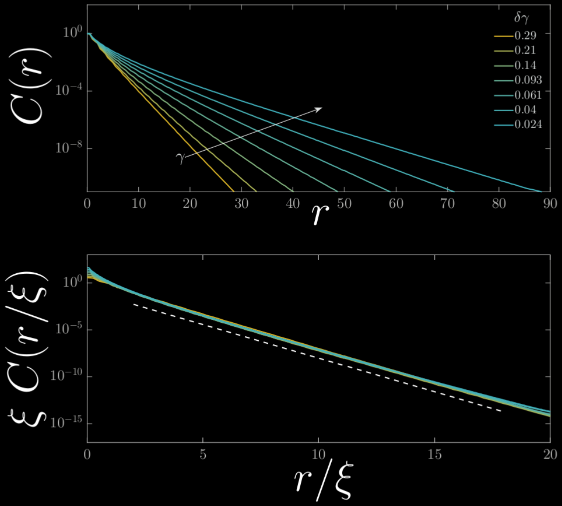

data measured at $\kappa \to 0^+$ and $\gamma < \gamma_c$

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Robbie Rens et al., PRE 2018

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Probing Local Force Propagation in Tensed Fibrous Gels

Shahar Goren[1,2,3], Maayan Levin[2,3], Guy Brand[2], Ayelet Lesman[1,3,*], and Raya Sorkin[2,3,*]

[1]School of Mechanical Engineering, The Iby and Aladar Fleischman Faculty of Engineering, Tel Aviv University, Israel
[2]School of Chemistry, Raymond & Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Israel, Israel
[3]Center for Chemistry and Physics of Living Systems, Tel Aviv University, Israel
[4]Center for Light-Matter Interaction, Tel Aviv University, Tel Aviv, Israel
[*]These authors jointly supervised this work
[*]correspondence to emails: ayeletlesman@tauex.tau.ac.il and rsorkin@tauex.tau.ac.il

April 25, 2022

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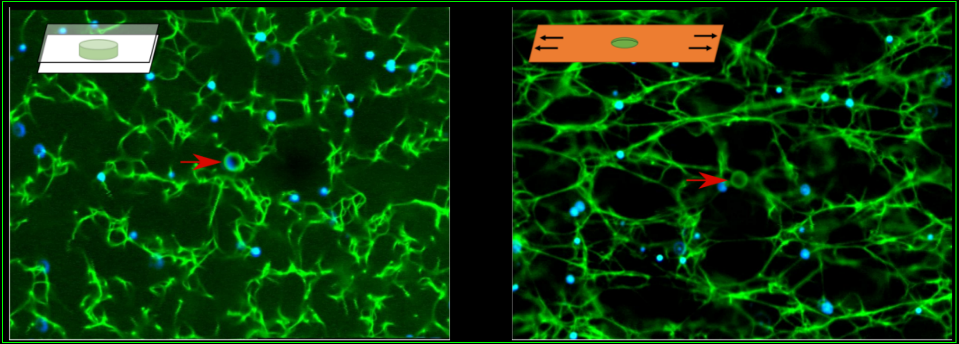

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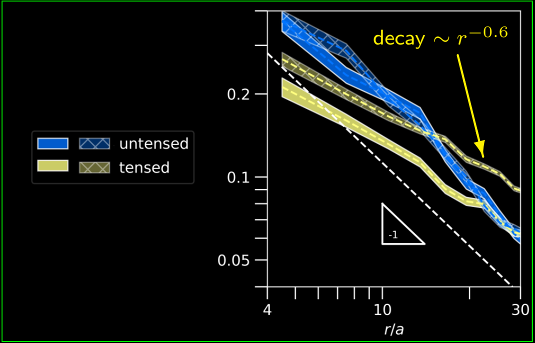

open questions

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EL and Eran Bouchbinder, arXiv:2209.04237

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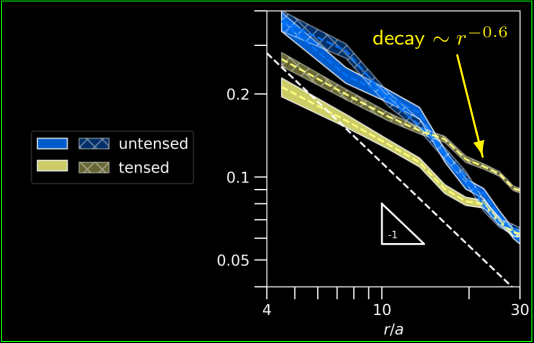

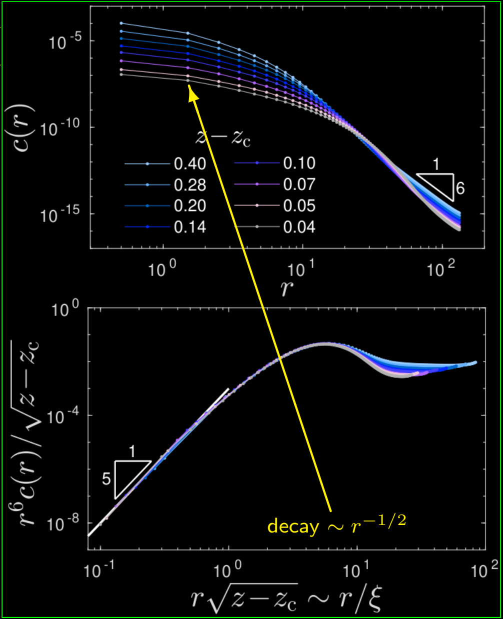

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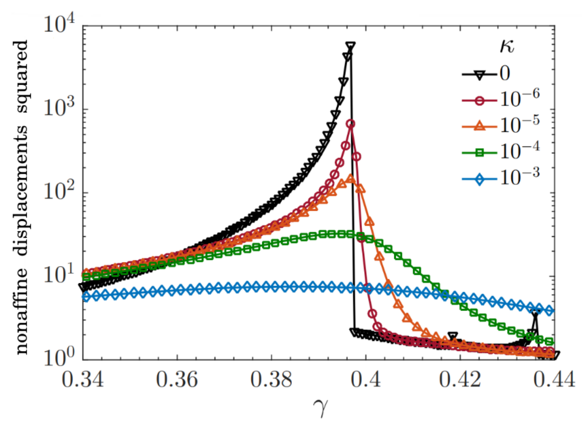

Robbie Rens PhD thesis 2019

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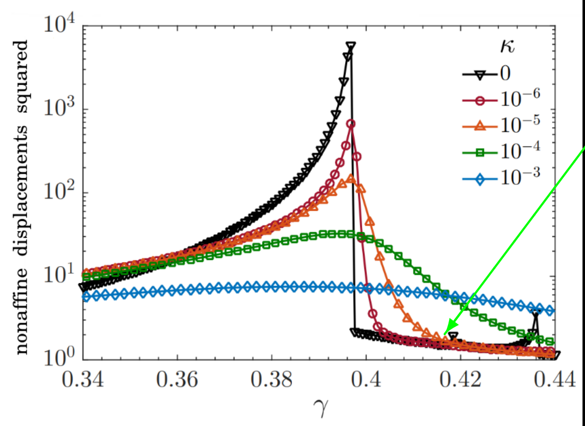

does **the same** $\delta\gamma_\star(\kappa)$ also hold above γ_c?

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open questions

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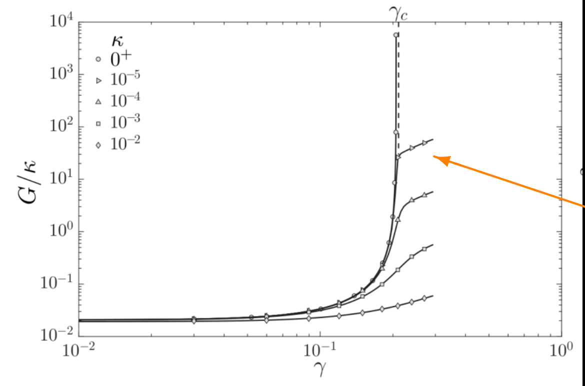

how does $G(\gamma)$ behave **above** $\gamma_c + \delta\gamma_\star(\kappa)$?

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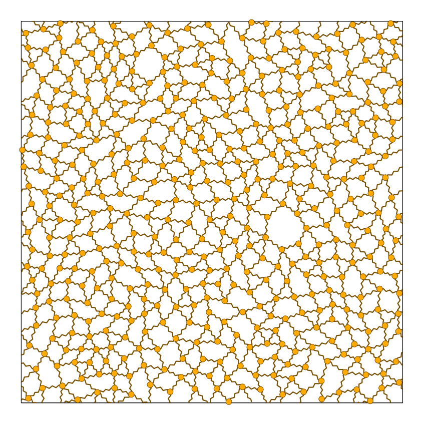
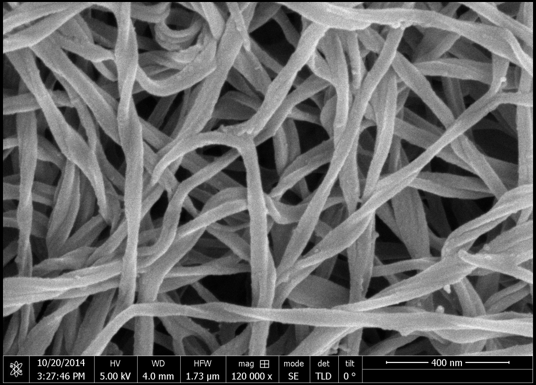
open questions

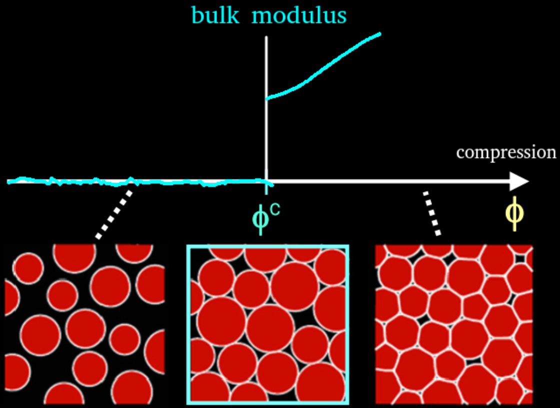
(iii) is our model too simple? Are we missing essential ingredients? Does 2D tell us about 3D?



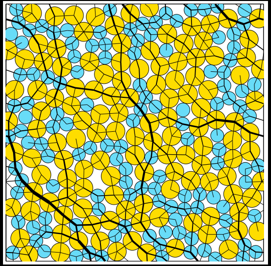
$$\overset{??}{=}$$

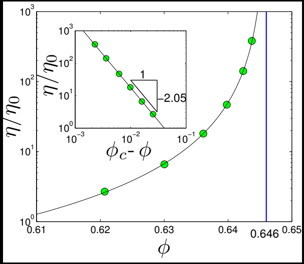


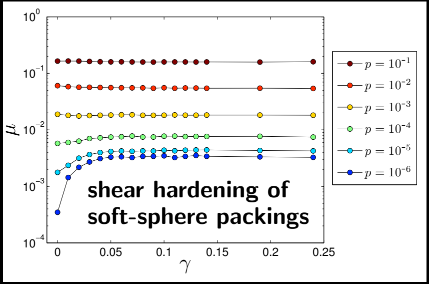
strain stiffening is a member of a set of 'jamming' problems:



non-Brownian suspension viscosity







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recall:

Wyart (phd thesis, 2005) showed that (for relaxed spring networks)

$$G = \frac{1}{V} \sum_{\text{states of self-stress } \varphi_\ell} \langle \varphi_\ell | \partial r / \partial \gamma \rangle^2$$

strain stiffening is a member of a set of ‘jamming’ problems.

common to all these problems is the **coupling** of the state-of-self-stress (or the minimal eigenmode of $\mathcal{S}\mathcal{S}^T$) **to the imposed deformation**

recall:

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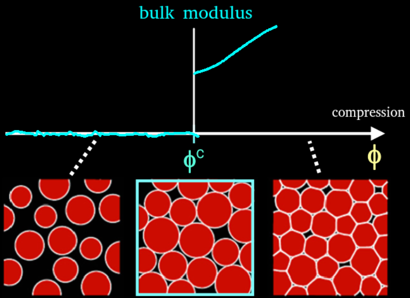
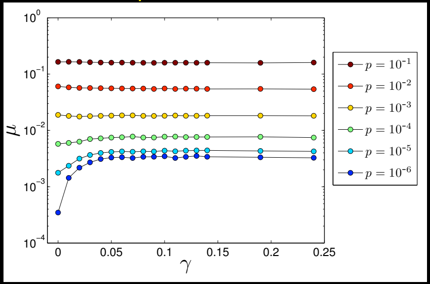
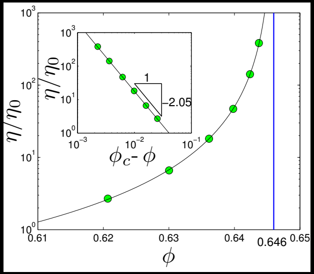


harmonic spheres under shear



non-Brownian suspension viscosity



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מכון ויצמן למדע

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further reading:

→ Gustavo Düring, EL, and Matthieu Wyart, *Length scales and self-organization in dense suspension flows*, PRE **89**, 022305 (2014)

→ Robbie Rens, Carlos Villarroel, Gustavo Düring, and EL, Micromechanical theory of strain-stiffening of biopolymer networks, PRE **98**, 062411 (2018)

→ EL and Eran Bouchbinder, *Scaling theory of critical strain-stiffening in athermal biopolymer networks*, arXiv:2208.08204.

→ EL and Eran Bouchbinder, *Anomalous elasticity of disordered networks*, arXiv:2209.04237.

thanks for your attention!